On class groups and Iwasawa modules of CM-fields

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Abstract

Our aim in this paper is to give a clear view of what can be proved on the Fitting ideal of the Pontryagin dual of the minus class group of a CM-field as a Galois module. The class group we study is the classical full class group, and not the ray class group mod T (we are particularly interested in the Teichmüller character component of the class group). We show that several theorems in the existing literature hold unconditionally, using recent groundbreaking results by Dasgupta, Kakde, Silliman, and Wang. In particular, we give a simple proof of the Equivariant Iwasawa main conjecture including the case $p = 2$, using their keystone theorems due to Dasgupta and Kakde in [Annals of Math.197(2023),289-388] and due to Dasgupta, Kakde, Silliman and Wang in [arXiv:2310.16399]. We unconditionally compute the Fitting ideal of the Pontryagin dual of the minus class group of the cyclotomic $\mathbb{Z}_p$-extension of a CM-field, and also the Fitting ideal of S-ramified Iwasawa modules for totally real number fields. The numbers of minimal generators of these Iwasawa modules are also studied.

1 Introduction

Suppose that K/k is a finite abelian extension of number fields with Galois group G. We study in this paper the full ideal class group Cl_K (the quotient of the group of non-zero fractional ideals by the subgroup of non-zero principal fractional ideals), which we regard as a G-module. Since we can decompose $\mathrm{Cl}_K = \bigoplus_p \mathrm{Cl}_K \otimes \mathbb{Z}_p$ into the p-components for each prime p, in order to understand Cl_K as a G-module, we may study each G-module $\mathrm{Cl}_K \otimes \mathbb{Z}_p$. In the following, we fix a prime number p and denote $\mathrm{Cl}_K \otimes \mathbb{Z}_p$ by A_K.

We are interested in the relationship between the class group and zeta values. Such a relationship can be seen most explicitly when K/k is a CM-extension, namely K is a CM-field and k is a totally real field, so we assume it.

We assume $p > 2$ in this Introduction, and denote by A_K^- the part on which the complex conjugation acts as -1. We study A_K^- as a $\mathbb{Z}_p[G]$-module. It has been gradually recognized that the Pontryagin dual $(A_K^-)^\vee$ of A_K^- is easier to handle than the module A_K^- itself (for example, see Greither [13], the appendix of the author [23], and [14]). Our objective in this paper is to give a clear view of what can be proved about the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_p[G]}((A_K^-)^\vee)$ of the Pontryagin dual of the class groups by combining recent results in various papers (for the Fitting ideal $\mathrm{Fitt}_R(M)$ of an R-module M, see [29]).

A recent groundbreaking result by Dasgupta and Kakde [8] completely determines the Fitting ideal of the dual of the ray class group mod T (which is also called T-smoothed class group recently). However, our interest in this paper is in the non T-smoothed class groups, namely the classical full class groups. We decompose $G = \Delta \times G_p$ where Δ is of order prime to p and G_p is a p-group. We have direct decomposition of A_K^- into χ-components for characters χ of Δ (see §2.3 (3)). Let ω be the Teichmüller character giving the action on the p-th roots of unity. If χ is an odd character such that $\chi \neq \omega$, then we know the Fitting ideal of $(A_K^\chi)^\vee$ as a G_p-module by the above theorem of Dasgupta and Kakde (see Theorem 2.4). So our main interest is in the ω-component for which the aspect is very different from the non-ω component. To get a good understanding of the ω-component, see the numerical example in §2.4. We determine the Fitting ideal of $(A_K^\omega)^\vee$, assuming certain conditions (see Corollary 3.6), but cannot determine it, in general. We consider the cyclotomic $\mathbb{Z}_p$-extension K_∞/K, and study the Fitting ideal of $(A_{K_\infty}^-)^\vee$ as a $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$-module. We determine it completely, including the ω-component (see Theorems 3.1, 3.4, 3.7).

In §5 we reduce the above problem on the ω-component to the problem on the Iwasawa modules over totally real number fields. For an abelian p-extension F/k of totally real number fields and the cyclotomic $\mathbb{Z}_p$-extension F_∞, we study in §5 the maximal abelian pro-p extension $M_{F_\infty, S_p \cup S_\infty}/F_\infty$ that is unramified outside p and ∞, and its Galois group $X_{F_\infty,p}$. We determine its Fitting ideal as a $\mathrm{Gal}(F_\infty/k)$-module unconditionally. To do this, we use the equivariant Iwasawa main conjecture (EIMC for short) for abelian extensions of totally real number fields. Johnston and Nickel prove EIMC in their paper [19], using the results of Dasgupta and Kakde in [8]. However, they assume $p > 2$, so we give in §4 a different and simple proof of EIMC including the case $p = 2$, using the keystone theorem of Dasgupta and Kakde [8] and of Dasgupta, Kakde, Silliman and Wang [10], which was a part of conjecture by Burns, Sano and the author in [5]. This implication is natural because the conjecture of Burns, Sano and the author in [5] can be regarded as a finite extension version of the Iwasawa main conjecture.

In this paper, we adopt a slightly different definition of the Stickelberger ideal, which is equivalent to the usual definition but more intuitive and easier to use.

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2 Full ideal class group

2.1 cyclotomic fields

As an introduction to the theory of the Galois action on ideal class groups, we consider the most classical and famous example, that is $k = \mathbb{Q}$ and $K = \mathbb{Q}(\mu_p)$, the cyclotomic field of p-th roots of unity for some odd prime p. Put $G = \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$. Let $\hat{G}$ be the group consisting of p-adic characters $\chi : G \to \mathbb{Q}_p^{\times}$ for $G = \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$. For any $\mathbb{Z}_p[G]$-module M and any $\chi \in \hat{G}$, we define $M(\chi) = \{x \in M \mid \sigma(x) = \chi(\sigma)x \text{ for all } \sigma \in G\}$, then M is decomposed into $M = \bigoplus_{\chi \in \hat{G}} M(\chi)$ where χ runs over all characters of G. Thus we have $A_K = \bigoplus_{\chi \in \hat{G}} A_K(\chi)$. In this article, we concentrate on the minus class group, so suppose χ is an *odd* character, namely $\chi(\rho) = -1$ where ρ is the complex conjugation in G. Let $\omega \in \hat{G}$ be the Teichmüller character giving the action on the group of p-th roots of unity. Then $\chi(\rho) = -1$ implies that we can write $\chi = \omega^i$ for odd i. We take i in the range $1 \leq i \leq p-2$. It is not difficult to check that $A_K^{\omega} = 0$ ([35] Proposition 6.16). For the rest of the i, the following famous theorem is proved by Mazur and Wiles ([28] Chap.I Theorem 2) as a corollary of the Iwasawa main conjecture by building on the efforts of many people including Herbrand and Ribet.

Theorem 2.1. (Mazur and Wiles) *Suppose that i is odd such that $1 < i \leq p-2$. Then for $K = \mathbb{Q}(\mu_p)$, we have*

$$\# A_K^{\omega^i} = \# \mathbb{Z}_p / L(0, \omega^{-i}) = \# \mathbb{Z}_p / B_{1,\omega^{-i}}.$$

Note that the generalized Bernoulli number $B_{1,\omega^{-i}} = \frac{1}{p} \sum_{a=1}^{p-1} \omega^{-i}(a) a$ is in $\mathbb{Z}_p$ in this range of i. Also, we know $B_{1,\omega^{-i}} \equiv B_{p-i}/(p-i) \pmod{p}$ where B_{p-i} is the $(p-i)$-th Bernoulli number. Thus the above theorem is a refinement of the famous theorem by Herbrand and Ribet

$$p \mid B_{p-i} \Longleftrightarrow A_{\mathbb{Q}(\mu_p)}^{\omega^i} \neq 0.$$

Since the Kubota-Leopoldt p-adic L-function for ω^{1-i} has no trivial zero, the above theorem is an immediate consequence of the Iwasawa main conjecture. In order to treat more general character χ, we encounter the difficulty coming from "trivial zeros".

2.2 semi-simple case

We consider a general abelian CM-extension K/k. Namely, we suppose that k is a totally real field, K is a CM-field, and K/k is a finite abelian extension with Galois group G. We first assume that G is of *order prime to p*. We denote by $\hat{G}$ the group of p-adic characters $\chi : G \to \overline{\mathbb{Q}}_p^{\times}$ where $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$. For $\chi_1, \chi_2 \in \hat{G}$, we define an equivalence relation $\sim$ by $\chi_1 \sim \chi_2 \Longleftrightarrow \chi_1 = \sigma\chi_2$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then since $\#G$ is prime to p, $\mathbb{Z}_p[G]$ is decomposed into a product of discrete valuation rings;

$$\mathbb{Z}_p[G] = \bigoplus_{\chi \in \hat{G}/\sim} O_\chi$$

where $O_\chi = \mathbb{Z}_p[\mathrm{Image}\,\chi]$ on which G acts via χ. Therefore, any $\mathbb{Z}_p[G]$-module M is also decomposed as

$$M = \bigoplus_{\chi \in \hat{G}/\sim} M^\chi$$

where $M^\chi = M \otimes_{\mathbb{Z}_p[G]} O_\chi$. Thus, in order to know the size of a G-module M, it suffices to know the orders of all the M^χ. We note that if $\#G$ divides $p-1$, the image of χ is in $\mathbb{Z}_p$, and M^χ is isomorphic to $M(\chi)$ defined in the previous subsection.

Since we assumed p is prime to $\#G$, p has to be odd. Then $A_K = A_K^+ \oplus A_K^-$ where $A_K^{\pm}$ is the $\pm$-eigenspace of the complex conjugation $\rho \in G$. We can decompose

$$A_K^- \simeq \bigoplus_{\substack{\chi \in \hat{\Delta}/\sim \\ \chi(\rho)=-1}} A_K^\chi$$

where χ runs over all equivalence classes of odd characters.

For an odd character χ of G such that $\chi \neq \omega$, we know by Deligne and Ribet [11] and Pierrette Cassou-Noguès [7] that $L(0, \chi^{-1})$ is in O_χ (and is nonzero) where $L(s, \chi)$ is the L-function of χ. For $\chi = \omega$, $\#\mu_{p^\infty}(K)L(0, \omega^{-1})$ is in $O_\omega = \mathbb{Z}_p$ where $\mu_{p^\infty}(K)$ is the group of p-power roots of unity in K.

Theorem 2.2. (Dasgupta and Kakde) *Suppose that χ is an odd character such that $\chi \neq \omega$. Then we have*

$$\#A_K^\chi = \#(O_\chi/(L(0,\chi^{-1}))). \tag{1}$$

For $\chi = \omega$, we get

$$\#A_K^\omega = \#(O_\omega/(\#\mu_{p^\infty}(K)L(0,\omega^{-1}))). \tag{2}$$

We will give a proof of this theorem in §2.5.

Remark 2.3. (1) If $\chi(\mathfrak{p}) = 1$ for some p-adic prime $\mathfrak{p}$ of k, the p-adic L-function $L_p(s,\chi^{-1}\omega)$ (of Deligne and Ribet) has a trivial zero at $s = 0$. In this case, the descent argument does not work well, and one cannot directly deduce Theorem 2.2 from the Iwasawa main conjecture. When $k = \mathbb{Q}$, Mazur and Wiles proved Theorem 2.2 by a detailed study of the descent argument ([28]). Some partial results were known before Dasgupta and Kakde (for example, if there is at most one p-adic prime such that $\chi(\mathfrak{p}) = 1$, then it was proved in [5] Cor. 1.9 that the conclusion of Theorem 2.2 holds), but a complete proof of Theorem 2.2 was first obtained through the work of Dasgupta and Kakde (see §2.5). They did not use the Iwasawa main conjecture in their proof, and studied directly the class group (unramified extensions) of a number field.
(2) The right hand sides of (1) and (2) are determined by χ and independent of K. Hence, the left hand sides of (1) and (2) should not depend on K either. This can be easily checked as follows. Let K_χ be the CM-field corresponding to $\operatorname{Ker}\chi$. Then the norm argument shows that the natural map $(A_{K_\chi})^\chi \xrightarrow{\simeq} (A_K)^\chi$ is bijective since $[K:k]$ is prime to p.

2.3 general case

Suppose now that K/k is a CM-extension, and $G = \mathrm{Gal}(K/k)$ is a general abelian group. From this subsection through §2.5, we assume $p > 2$. We write G as $G = \Delta \times G_p$ where Δ is of order prime to p and G_p is a p-group. Since $\mathbb{Z}_p[\Delta]$ is decomposed into $\mathbb{Z}_p[\Delta] = \bigoplus_{\chi\in\hat{\Delta}/\sim} O_\chi$, we have

$$\mathbb{Z}_p[G] = \bigoplus_{\chi\in\hat{\Delta}/\sim} O_\chi[G_p].$$

Therefore, any $\mathbb{Z}_p[G]$-module M is also decomposed as

$$M = \bigoplus_{\chi\in\hat{\Delta}/\sim} M^\chi$$

where $M^\chi = M \otimes_{\mathbb{Z}_p[\Delta]} O_\chi$, which is an $O_\chi[G_p]$-module.

Recall that we are assuming $p > 2$. We study

$$A_K^- = \bigoplus_{\substack{\chi \in \hat{\Delta}/\sim \\ \chi(\rho)=-1}} A_K^\chi. \tag{3}$$

Note that this is decomposition with respect to the action of Δ, not of G. So no information is lost in the above decomposition.

To understand a $\mathbb{Z}_p[G]$-module, we study its Fitting ideal (see [29] for the Fitting ideal $\mathrm{Fitt}_R(M)$ of an R-module M) instead of its order. Note that if R is a discrete valuation ring with finite residue field and M is a finitely generated torsion R-module, then we have $\#(R/\mathrm{Fitt}_R(M)) = \#M$.

As we mentioned in §1, the Pontryagin dual $(A_K^-)^\vee$ of A_K^- has better properties than the module A_K^- itself when we compute the Fitting ideals (see [13], [23], [14]). However, we note here that Atsuta and Kataoka in their recent paper [1] determine the Fitting ideal of the (T-smoothed) class group $(A_K^T)^-$ (not the dual), assuming the equivariant Tamagawa number conjecture for the minus part of K/k and $\mathbb{G}_m$. Since the equivariant Tamagawa number conjecture in this case was recently proved by Bullack, Burns, Daoud, and Seo [2] and also by Dasgupta, Kakde, and Silliman [9], the result by Atsuta and Kataoka is now unconditional. The shape of the Fitting ideal of $(A_K^T)^-$ is more complicated than that of $((A_K^T)^-)^\vee$.

We go back to our problem. Our objective is to know $\mathrm{Fitt}_{\mathbb{Z}_p[G]}((A_K^-)^\vee)$. To do this, it is enough to determine $\mathrm{Fitt}_{O_\chi[G_p]}((A_K^\chi)^\vee)$ for all odd $\chi \in \hat{\Delta}$ by the above decomposition.

We first define the Stickelberger ideals. For a finite abelian extension M/k and a finite set S of primes of k, we define the Stickelberger element $\theta_{M/k,S}$ as follows. We define

$$\theta_{M/k,S}(s) = \prod_{\psi \in \hat{\mathrm{Gal}}(M/k)} L_S(s, \psi^{-1})\epsilon_\psi$$

where $L_S(s, \psi^{-1})$ is the S-truncated L-function for ψ^{-1}, and

$$\epsilon_\psi = \frac{1}{\#\,\mathrm{Gal}(M/k)} \sum_{\sigma \in \mathrm{Gal}(M/k)} \psi(\sigma)\sigma^{-1}.$$

We define $\theta_{M/k,S} = \theta_{M/k,S}(0)$, which is known to be in $\mathbb{Q}[\mathrm{Gal}(M/k)]$ by Klingen and Siegel. We denote by $S_{\mathrm{ram}}(M/k)$ the set of all places of k

ramifying in M/k. If S contains $S_{\mathrm{ram}}(M/k)$, we also know by Deligne and Ribet [11] and Pierrette Cassou-Noguès [7] that

$$\mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(M/k)]}(\mu(M))\theta_{M/k,S} \subset \mathbb{Z}[\mathrm{Gal}(M/k)] \tag{4}$$

where $\mathrm{Ann}_{\mathbb{Z}[\mathrm{Gal}(M/k)]}(\mu(M))$ is the annihilator ideal of the group $\mu(M)$ of roots of unity in M.

When $S = S_{\mathrm{ram}}(M/k)$, we simply write $\theta_{M/k}$ for $\theta_{M/k,S_{\mathrm{ram}}(M/k)}$.

Going back to our CM-extension K/k, we denote by S_∞ the set of infinite places of k. For $v \in S_{\mathrm{ram}}(K/k)\backslash S_\infty$, let I_v be the inertia group of v in G, and $N_{I_v} = \Sigma_{\sigma\in I_v}\sigma$ the norm element of I_v. For any subset J of $S_{\mathrm{ram}}(K/k)\setminus S_\infty$, let K_J be the fixed subfield of the subgroup of G generated by I_v for all $v \in J$ (so K_J/k is the maximal subextension of K/k that is unramified at all primes in J), and put $N_J = \prod_{v\in J} N_{I_v} \in \mathbb{Z}[G]$. If J is empty, we define $K_J = K$ and $N_J = 1$. Then the multiplication by N_J defines a homomorphism

$$N_J : \mathbb{Q}[\mathrm{Gal}(K_J/k)] \longrightarrow \mathbb{Q}[G],$$

which we also denote by the same letter N_J. This is not a norm homomorphism for K/K_J but the multiplication by some constant of the norm homomorphism. We define $\Theta(K/k)$ to be the $\mathbb{Z}[G]$-module in $\mathbb{Q}[G]$ generated by

$$\{N_J(\theta_{K_J/k,S_{\mathrm{ram}}(K/k)\backslash J}) \mid J \subset S_{\mathrm{ram}}(K/k)\setminus S_\infty\}.$$

An alternative definition of this module $\Theta(K/k)$ is as follows. Put $U_v = (N_{I_v}, 1 - \frac{N_{I_v}}{\# I_v}\mathrm{Frob}_v^{-1})\,\mathbb{Z}[G] \subset \mathbb{Q}[G]$ where Frob_v is the Frobenius of v in G, and $w = \sum_{\psi\in\hat{G}} L(0,\psi^{-1})\epsilon_\psi \in \mathbb{Q}[G]$. The difference between w and $\theta_{K/k,S}$ is that the L-functions appearing in the definition of $\theta_{K/k,S}$ are S-imprimitive. Then

$$\Theta(K/k) = (\prod_{v\in S_{\mathrm{ram}}(K/k)\backslash S_\infty} U_v)w \subset \mathbb{Q}[G].$$

The above equality is proved in Proposition 3.1 in [25]. We adopt the first definition of $\Theta(K/k)$ in this article because it is more intuitive, and also useful below. We define

$$\Theta(K/k)_p = (\Theta(K/k)\otimes\mathbb{Z}_p)^- \subset \mathbb{Q}_p[G]^-.$$

Consider the decomposition $\mathbb{Q}_p[G] = \bigoplus_{\chi\in\hat{\Delta}/\sim} O_\chi[1/p][G_p]$. For an element $x \in \mathbb{Q}_p[G]$, we write $x = (x^\chi)_\chi$ where x^χ is the χ-component in $O_\chi[1/p][G_p]$. For a $\mathbb{Z}_p[G]$-module M, we also denote by M^χ the χ-component of M.

We assume that $\chi \neq \omega$, and consider $\Theta(K/k)_p^\chi \subset O_\chi[1/p][G_p]$. Then, by the above-mentioned property (4) that Deligne and Ribet and Cassou-Noguès proved, we have

$$N_J(\theta_{K_J/k,S_{\mathrm{ram}}(K/k)\setminus J})^\chi \in O_\chi[G_p].$$

It follows from the definition of $\Theta(K/k)$ that $\Theta(K/k)_p^\chi \subset O_\chi[G_p]$.

For any group ring $R[G]$ we denote by $x \mapsto x^\#$ the involution $R[G] \longrightarrow R[G]$ induced by $\sigma \mapsto \sigma^{-1}$ for all $\sigma \in G$. Greither in [13] proved the following theorem, assuming the equivariant Tamagawa number conjecture. We will explain in §2.5 that the main theorem of Dasgupta and Kakde in [8] unconditionally implies the following theorem.

Theorem 2.4. (Dasgupta and Kakde) *Suppose that χ is an odd character of Δ with $\chi \neq \omega$. Then we have*

$$\mathrm{Fitt}_{O_\chi[G_p]}((A_K^\chi)^\vee) = (\Theta(K/k)_p^\chi)^\#.$$

In the simplest setting that $G_p = 1$, the above equality becomes

$$\mathrm{Fitt}_{O_\chi}(A_K^\chi) = \mathrm{Fitt}_{O_\chi}((A_K^\chi)^\vee) = \theta_{K/k,S}^\chi O_\chi = L(0, \chi^{-1})O_\chi$$

where $S = S_{\mathrm{ram}}(K/k)$. Thus Theorem 2.4 implies Theorem 2.2 (1) in the previous subsection.

Theorem 2.2 can be generalized to a more general order character.

Corollary 2.5. *Suppose that ψ is an odd character of G with $\psi_{|\Delta} \neq \omega$. Let K_ψ be the subfield of K such that $\mathrm{Gal}(K_\psi/k)$ is isomorphic to the image of ψ by ψ. We put $(A_{K_\psi})_\psi = (A_{K_\psi}) \otimes_{\mathbb{Z}_p[\mathrm{Gal}(K_\psi/k)]} O_\psi$ where $O_\psi = \mathbb{Z}_p[\mathrm{Image}\,\psi]$ on which the Galois group acts via ψ. Then we have*

$$\#(A_{K_\psi})_\psi = \# O_\psi/(L(0, \psi^{-1})).$$

In the case that p divides the order of ψ, $(A_K)_\psi$ is different from $(A_{K_\psi})_\psi$, in general, and we have to consider K_ψ to get the above formula.

When $k = \mathbb{Q}$, the above corollary was proved by D. Solomon in [33].

Proof. Put $M = K_\psi$. We apply Theorem 2.4 to the extension M/k. We write $\mathrm{Gal}(M/k) = \Delta_M \times (G_M)_p$. Put $\chi = \psi_{|\Delta_M}$. Then $\chi \neq \omega$ by our assumption. By the definition of $\Theta(M/k)_p^\chi$, the image of $\Theta(M/k)_p^\chi$ in $O_\chi[(G_M)_p]$ under the homomorphism $\psi : O_\chi[(G_M)_p] \to O_\psi$ induced by ψ, is generated by $\psi(\theta_{M/k,S})$ where $S = S_{\mathrm{ram}}(M/k)$. This shows that

$$\begin{aligned} \mathrm{Fitt}_{O_\psi}((A_M)_\psi) &= \mathrm{Fitt}_{O_\psi}(((A_M)_\psi)^\vee) = \psi(\mathrm{Fitt}_{O_\chi[G_p]}((A_M^\chi)^\vee)) \\ &= \psi(\theta_{M/k,S})O_\psi = L(0, \psi^{-1})O_\psi. \end{aligned}$$

□

2.4 ω-component

Let ω be the Teichmüller character. By what we explained in the previous subsection, it suffices to determine $\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee)$ to know $\mathrm{Fitt}_{\mathbb{Z}_p[G]}((A_K^-)^\vee)$.

In this subsection, to give a good understanding of the ω-component, we provide a numerical example. Take

$$k = \mathbb{Q}(\sqrt{1901}), \quad K = k(\sqrt{-3}, \alpha, \beta) \ \text{ and } p = 3$$

where α and β satisfy $\alpha^3 - 84\alpha - 191 = 0$ and $\beta^3 - 57\beta - 68 = 0$. Then $G = \mathrm{Gal}(K/k) = \Delta \times G_p = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$. A machine computation shows that

$$A_K^- = A_K^\omega = \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^{\oplus 6},$$

and we can compute the action of G on it (see [27] §2 for the Galois action).

Put $S_{\mathrm{ram}} = S_{\mathrm{ram}}(K/k)$. Since 3 is inert in $k/\mathbb{Q}$ and totally ramified in $k(\beta)/k$, 3 is in S_{ram}. Also, $k(\alpha)/k$ is unramified everywhere, and $k(\beta)/k$ is unramified outside 3. Therefore, S_{ram} consists of 3 and two infinite places ($\# S_{\mathrm{ram}} = 3$).

For $p = 3$, we compute the Fitting ideal of the dual of the class group to get

$$\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee) = \mathfrak{m}^2\, \theta^{\#}_{K/k, S_{\mathrm{ram}}} = \mathfrak{m}^2\, \theta^{\#}_{K/k} \tag{5}$$

where $\mathfrak{m}$ is the maximal ideal of the local ring $O_\omega[G_p]$ (see [15] §4 Page 962).

For this numerical example, it is easy to compute $\Theta(K/k)_p^\omega = \theta_{K/k}^\omega O_\omega[G_p] \subset \mathbb{Q}_p[G_p]$. Also, it is easy to check $\mathrm{Ann}_{O_\omega[G_p]}(\mu(K)) = \mathrm{Ann}_{O_\omega[G_p]}(\mu(K)) = \mathfrak{m}$. These computations show that

$$\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee) = \mathfrak{m}^2\, \theta^{\#}_{K/k} \neq (\mathrm{Ann}_{O_\omega[G_p]}(\mu(K))\Theta(K/k)_p)^{\omega,\#} = \mathfrak{m}\, \theta^{\#}_{K/k}.$$

This example shows that $(\mathrm{Ann}_{O_\omega[G_p]}(\mu(K))\Theta(K/k)_p)^{\#}$ is not in $\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee)$, and that a simple guess $\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee) = (\mathrm{Ann}_{O_\omega[G_p]}(\mu(K))\Theta(K/k)_p)^{\omega,\#}$ does not hold, in general.

We will later give a general theorem (see Corollary 3.6), which implies (5) theoretically. This Corollary 3.6 describes $\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee)$ under certain assumptions. However, the author does not know a description in a general setting.

Problem 2.6. *For a general abelian CM-extension K/k such that $\mu_p \subset K^\times$, describe $\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee)$ using Stickelberger elements.*

Remark 2.7. For $k = \mathbb{Q}$ and $K = \mathbb{Q}(\mu_n)$, the cyclotomic field of n-th roots of unity for some n (or more generally, when K satisfies the condition (A_p) in [22] §3.1), we can show that

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(A_K^-) = \Theta(K/k)_p \cap \mathbb{Z}_p[G]^-$$

(see [26]). However, even for $k = \mathbb{Q}$ and $K = \mathbb{Q}(\mu_n)$ with certain n, we have

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A_K^-)^\vee) \neq \Theta(K/k)_p^\# \cap \mathbb{Z}_p[G]^-.$$

We can see this fact by studying the ω-complonent $(A_K^\omega)^\vee$ though we do not give here the details.

2.5 T-smoothing

In this subsection we introduce the main theorem of Dasgupta and Kakde in [8], and deduce theorems in the subsections §2.2 and 2.3 from it.

Let S, T be finite sets of places of k such that $S \supset S_\infty$ and $S \cap T = \emptyset$.

For any finite abelian CM-extension K/k, the S-truncated equivariant zeta function $\theta_{K/k,S}(s)$ was defined in §2.3. We define

$$\theta_{M/k,S}^T(s) = \prod_{\psi \in \hat{\mathrm{Gal}}(M/k)} (\prod_{v \in T} (1 - \psi(v)^{-1} N(v)^{1-s})) L_S(s, \psi^{-1}) \epsilon_\psi$$

and the (S,T)-Stickelberger element by $\theta_{K/k,S}^T = \theta_{K/k,S}^T(0)$. Therefore, if T consists of primes which are unramified in K, we have

$$\theta_{K/k,S}^T = \theta_{K/k,S}^T(0) = (\prod_{v \in T} (1 - \mathrm{Frob}_v^{-1} N(v))) \theta_{K/k,S}(0). \tag{6}$$

Thus, if $S \supset S_{\mathrm{ram}}(K/k)$ and T is non-empty, by the property (4) we have $\theta_{K/k,S}^T \in \mathbb{Z}[\mathrm{Gal}(K/k)]$. We simply write $\theta_{K/k}^T$ for $\theta_{K/k,S_{\mathrm{ram}}(M/k)}^T$.

Now we define the T-smoothed Stickelberger ideal $\Theta(K/k)^T$ to be the ideal of $\mathbb{Z}[\mathrm{Gal}(K/k)]$ generated by

$$\{N_J(\theta_{K_J/k, S_{\mathrm{ram}}(K/k) \setminus J}^T) \mid J \subset S_{\mathrm{ram}}(K/k) \setminus S_\infty\}$$

where N_J, K_J are as in §2.3. This ideal is called the Sinnott-Kurihara ideal ([8] page 295). We put $\Theta(K/k)_p^T = (\Theta(K/k)^T \otimes \mathbb{Z}_p)^-$.

We denote by $\mathcal{O}_{K,S}$ the ring of S-integers (integral elements outside S) of K. Let T_K be the set of primes of K above T. We define the (S,T)-unit group by $(\mathcal{O}_{K,S}^T)^\times = \{x \in \mathcal{O}_{K,S}^\times \mid x \equiv 1 \pmod{w} \text{ for all } w \in T_K\}$, and define $\mathrm{Cl}^T_{K,S}$ to be the ray class group of $\mathcal{O}_{K,S}$ modulo $\Pi_{w\in T_K} w$. We assume that $(\mathcal{O}^T_{K,S})^\times$ is $\mathbb{Z}$-torsion free. This condition is mild, and is satisfied, for example, if T contains at least two primes of different residue characteristics.

When $S = S_\infty$, we write $(\mathcal{O}^T_K)^\times$ for $(\mathcal{O}^T_{K,S_\infty})^\times$, and Cl^T_K for $\mathrm{Cl}^T_{K,S_\infty}$. We fix a prime p and define $A^T_K = \mathrm{Cl}^T_K \otimes \mathbb{Z}_p$. As in §2.3, we consider the Pontryagin dual $((A^T_K)^-)^\vee$. Recall that $G = \mathrm{Gal}(K/k)$.

The next theorem was conjectured by the author ([25] Conjecture 3.2) and proved by Dasgupta and Kakde ([8] Theorem 3.5) as the main theorem of [8].

Theorem 2.8. (Dasgupta and Kakde [8] Theorem 3.5) *Suppose that $(\mathcal{O}^T_K)^\times$ is $\mathbb{Z}$-torsion free, and that p is odd. Then we have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(((A^T_K)^-)^\vee) = (\Theta(K/k)^T_p)^\#.$$

We first state the immediate consequences of this groundbreaking theorem.

Corollary 2.9. (1) (Strong Brumer-Stark conjecture, [8] Theorem 1.3) *We have $(\theta^T_{K/k})^\# \in \mathrm{Fitt}_{\mathbb{Z}_p[G]}((A^T_K)^\vee)$.*

(2) *If G_p is cyclic, we have*

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))\theta_{K/k} \subset \mathrm{Fitt}_{\mathbb{Z}_p[G]}(A_K)$$

and

$$(\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))\theta_{K/k})^\# \subset \mathrm{Fitt}_{\mathbb{Z}_p[G]}((A_K)^\vee).$$

Proof. Since $\theta^T_{K/k}$ is in $\Theta(K/k)^T$ by definition, Theorem 2.8 implies (1) by noting that $(\theta^T_{K/k})^+ = 0$.

Suppose that G_p is cyclic. Then for any finite $\mathbb{Z}_p[G]$-module M, we have $\mathrm{Fitt}_{\mathbb{Z}_p[G]}(M)^\# = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(M^\vee)$ (cf. [28] Appendix Proposition 1). Therefore, we have $\theta^T_{K/k} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]}(A^T_K)$. Since the natural homomorphism $A^T_K \to A_K$ is surjective, this implies $\theta^T_{K/k} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]}(A_K)$. This holds for any T, which implies the first inclusion of (2) by Tate [34] Chap. IV Lemme 1.1. We get the second inclusion of (2), using the first inclusion and the formula $\mathrm{Fitt}_{\mathbb{Z}_p[G]}(M)^\# = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(M^\vee)$ again. □

Remark 2.10. The above strong Brumer-Stark conjecture is stronger than the Brumer-Stark conjecture, which predicts that $\theta^T_{K/k} \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(A^T_K)$ because we now have

$$\theta^T_{K/k} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]}((A^T_K)^\vee)^\# \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}((A^T_K)^\vee)^\# = \mathrm{Ann}_{\mathbb{Z}_p[G]}(A^T_K)$$

and the Fitting ideal is smaller than the annihilator, in general.

We will show that Theorem 2.8 implies Theorem 2.4. Suppose that $\chi \in \hat{\Delta}$ satisfies $\chi \neq \omega$. Then we can take T such that $N(v) \not\equiv \chi(\mathrm{Frob}_v) \pmod{p}$ for all $v \in T$ where $N(v)$ is the norm of v. For $w \in T_K$, let $\kappa(w)$ be the residue field of w. Since $(\bigoplus_{w\in T_K} \kappa(w)^\times \otimes \mathbb{Z}_p)^\chi = 0$ by our choice of T, we have $(A_K^T)^\chi = A_K^\chi$. Also, since $(\theta^T_{F_J/k,S_{\mathrm{ram}}(K/k)\setminus J}/\theta_{F_J/k,S_{\mathrm{ram}}(K/k)\setminus J})^\chi$ is a unit of O_χ, we have $(\Theta(K/k)_p^T)^\chi = \Theta(K/k)_p^\chi$. Thus we get Theorem 2.4.

As we explained after Theorem 2.4, Theorem 2.4 implies Theorem 2.2 (1). Next, we consider Theorem 2.2 (2). We may assume $K = K_\omega = k(\mu_p)$. For simplicity, using the Chebotarev density theorem, we take non-empty T such that every $v \in T$ splits completely in K and $\mathrm{ord}_p(\#\mu_{p^\infty}(K)) = \mathrm{ord}_p(N(v) - 1)$. From the exact sequence

$$0 \to \mu_{p^\infty}(K) \to (\bigoplus_{w\in T_K} \kappa(w)^\times \otimes \mathbb{Z}_p)^\omega \to ((A_K^T)^-)^\omega \to A_K^\omega \to 0,$$

we know that

$$\mathrm{ord}_p(\#A_K^\omega) = \mathrm{ord}_p(\#((A_K^T)^-)^\omega) - (\#T - 1)\,\mathrm{ord}_p(\#\mu_{p^\infty}(K)).$$

Since $\omega(\mathrm{Frob}_v) = 1$ for v in T, Theorem 2.8 implies

$$\mathrm{ord}_p(\#((A_K^T)^-)^\omega) = \mathrm{ord}_p((\theta^T_{K/k})^\omega) = \#T\,\mathrm{ord}_p(N(v) - 1) + \mathrm{ord}_p(\theta^\omega_{K/k}).$$

It follows that

$$\mathrm{ord}_p(\#A_K^\omega) = \mathrm{ord}_p(\theta^\omega_{K/k}) + \mathrm{ord}_p(\#\mu_{p^\infty}(K)) = \mathrm{ord}_p(L(0,\omega^{-1})\#\mu_{p^\infty}(K)),$$

which completes the proof of Theorem 2.2 (2).

2.6 Selmer modules

In this subsection, following [5] by Burns, Sano, and the author, we introduce two Selmer modules, which we will use in §4. Let K/k be a finite extension of number fields, S, T be finite disjoint sets of places of k such that $S \supset S_\infty$. We write S_K, T_K the set of places of K above S, T, respectively. We define (see [5] Definition 2.1)

$$(K^T)^\times = \{x \in K^\times \,|\, \mathrm{ord}_w(x-1) > 0 \text{ for all } w \in T_K\}$$

and

$$\mathrm{Sel}_S^T(K) = \mathrm{Coker}(\prod_{w\notin S_K\cup T_K} \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}((K^T)^\times, \mathbb{Z}))$$

where the above map is defined by $(x_w)_w \mapsto (a \mapsto \sum_w \mathrm{ord}_w(a)x_w)$.

Let $(\mathcal{O}_{K,S}^T)^\times$ be the (S,T)-unit group, and $\mathrm{Cl}_{K,S}^T$ the ray class group of $\mathcal{O}_{K,S}$ modulo $\prod_{w \in T_K} w$ as in the previous subsection. Then we have an exact sequence

$$0 \to \mathrm{Cl}_S^T(K)^\vee \to \mathrm{Sel}_S^T(K) \to \mathrm{Hom}_{\mathbb{Z}}((\mathcal{O}_{K,S}^T)^\times, \mathbb{Z}) \to 0$$

(Proposition 2.2 in [5]) where $\mathrm{Cl}_S^T(K)^\vee$ is the Pontryagin dual of $\mathrm{Cl}_S^T(K)$.

If $T = \emptyset$ is the empty set, we write $\mathrm{Sel}_S(K)$ for $\mathrm{Sel}_S^\emptyset(K)$. The following proposition which we use in §4 shows a relationship between $\mathrm{Sel}_S(K)$ and the usual discrete Selmer group in $H^1(K, \mathbb{Q}/\mathbb{Z}(1))$.

Proposition 2.11. *We have an isomorphism*

$$\mathrm{Hom}(\mathrm{Sel}_S(K), \mathbb{Q}/\mathbb{Z}) \simeq \{x \in K^\times \otimes \mathbb{Q}/\mathbb{Z} \,|\, \iota_v(x) \in U_v \otimes \mathbb{Q}/\mathbb{Z} \text{ for all } v \notin S_K\}$$

where $\iota_v : K^\times \otimes \mathbb{Q}/\mathbb{Z} \to K_v^\times \otimes \mathbb{Q}/\mathbb{Z}$ is the map induced by the natural inclusion $K \to K_v$, and U_v is the unit group of K_v.

Proof. Taking $\mathrm{Hom}(\,*\,, \mathbb{Q}/\mathbb{Z})$ of the exact sequence

$$\prod_{v \notin S_K} \mathbb{Z} \to \mathrm{Hom}(K^\times, \mathbb{Z}) \to \mathrm{Sel}_S(K) \to 0,$$

we have an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Sel}_S(K), \mathbb{Q}/\mathbb{Z}) \to K^\times \otimes \mathbb{Q}/\mathbb{Z} \to \bigoplus_{v \notin S_K} (K_v^\times / U_v) \otimes \mathbb{Q}/\mathbb{Z},$$

which completes the proof. □

Another Selmer module is the Ritter-Weiss module $\nabla_S^T(K)$, whose definition we do not describe here. One can define this module, using the global and local fundamental classes (see [30], [25] §2 and [8] Appendix A), and can also define it as the cohomology of the "Weil-étale cohomology complex" as in [5] Definition 2.6. This module sits in an exact sequence

$$0 \to \mathrm{Cl}_S^T(K) \to \nabla_S^T(K) \to x_{K,S} \to 0$$

where $x_{K,S} = \mathrm{Ker}(\bigoplus_{w \in S_K} \mathbb{Z} \to \mathbb{Z})$ (see Remark 2.7 in [5]).

Suppose that K/k is a finite abelian CM-extension with Galois group G. We assume that $(\mathcal{O}_{K,S}^T)^\times$ is $\mathbb{Z}$-torsion free. If S contains $S_{\mathrm{ram}}(K/k)$, both $\mathrm{Sel}_S^T(K)$ and $\nabla_S^T(K)$ have a quadratic presentation as $\mathbb{Z}[G]$-modules, namely there is an exact sequence $M_1 \to M_2 \to \mathrm{Sel}_S^T(K) \to 0$ of finitely generated $\mathbb{Z}[G]$-modules where M_1, M_2 are free $\mathbb{Z}[G]$-modules of the same rank, and the same for $\nabla_S^T(K)$ (see [5] Proposition 2.4 (iv)). Therefore, their Fitting ideals as $\mathbb{Z}[G]$-modules are principal. A main conjecture in [5] is [5] Conjecture 7.3. For a CM-extension K/k we consider here, it says that the Fitting ideals of these two modules are generated by $(\theta_{K/k,S}^T)^{\#}$ and $\theta_{K/k,S}^T$, respectively, though Conjecture 7.3 in [5] treats general abelian extensions of number fields.

Let p be an arbitrary prime number (p may be 2). We put $\mathrm{Sel}_S^T(K)_p = \mathrm{Sel}_S^T \otimes \mathbb{Z}_p$ and $\nabla_S^T(K)_p = \nabla_S^T(K) \otimes \mathbb{Z}_p$. For a $\mathbb{Z}_p[G]$-module M, we define M_- by $M/(1+\rho)M$ where ρ is the complex conjugation in G. Then M_- is a $\mathbb{Z}_p[G]_- = \mathbb{Z}_p[G]/(1+\rho)$-module. If $p > 2$, M_- coincides with the (-1)-eigenspace M^- of ρ, which we used in §§2.2, 2.3.

We will use in §4 the following theorem, which was proved by Dasgupta and Kakde in [8] for $p > 2$, and by Dasgupta, Kakde, Silliman, and Wang in [10] for $p = 2$ (we also need an argument in the proof of Lemma 6 in [9]).

Theorem 2.12. *(Dasgupta, Kakde, Silliman, Wang) Suppose that S contains $S_{\mathrm{ram}}(K/k)$ and $(\mathcal{O}_{K,S}^T)^\times$ is $\mathbb{Z}$-torsion free. Then we have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\nabla_S^T(K)_{p,-}) = \theta_{K/k,S}^T \, \mathbb{Z}_p[G]_-, \quad \textit{and}$$

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\mathrm{Sel}_S^T(K)_{p,-}) = (\theta_{K/k,S}^T)^{\#} \, \mathbb{Z}_p[G]_- \ .$$

We first note that the two equations are equivalent. In fact, if $A = (a_{ij})$ is a relation matrix of $\nabla_S^T(K)_p$ as a $\mathbb{Z}_p[G]$-module, $({}^tA)^{\#} = (a_{ji}^{\#})$ becomes a relation matrix of $\mathrm{Sel}_S^T(K)_p$ (see the proof of Lemma 2.8 in [5]). Therefore, we have only to prove the first equation.

We note that Dasgupta, Kakde, Silliman, and Wang proved a more delicate theorem as a keystone theorem in their proof of the (strong) Brumer-Stark conjecture. Let S_p be the set of all p-adic places. We put $\Sigma = (S_{\mathrm{ram}}(K/k) \cap S_p) \cup S_\infty$ and $\Sigma_0' = S_{\mathrm{ram}}(K/k) \setminus \Sigma$, and $\Sigma' = T \cup \Sigma_0'$. The theorem which was proved by Dasgupta and Kakde in [8] Theorem 3.3 for $p > 2$, and by Dasgupta, Kakde, Silliman, and Wang in [10] Theorem 1.2 for $p = 2$, is

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\nabla_\Sigma^{\Sigma'}(K)_{p,-}) = \theta_{K/k,\Sigma}^{\Sigma'} \, \mathbb{Z}_p[G]_- \ , \tag{7}$$

which they called the keystone theorem. By the argument in the proof of Lemma 6 in [9], we can change (Σ, Σ') to (S, T) to get the first equation in Theorem 2.12.

We also note that the minus part of the equivariant Tamagawa number conjecture for K/k, which is proved in [2] ($p > 2$) and [9], implies Theorem 2.12 by Theorem 7.5 in [5].

We remark that in [10] they compute $\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\nabla_{\Sigma}^{\Sigma'}(K)_{p,-})$ without assuming $S \supset S_\infty$.

3 Iwasawa modules for CM-fields

We assume that p is odd in this section. Let K/k be a finite abelian CM-extension with Galois group G. What we saw in §2 was that it suffices to compute $\mathrm{Fitt}_{O_\omega[G_p]}((A_K^\omega)^\vee)$ in order to compute $\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A_K^-)^\vee)$.

This is difficult in general, so we study a relatively easier problem, namely consider the class group of the cyclotomic $\mathbb{Z}_p$-extension of K rather than the class group of K of finite degree. Let K_∞/K be the cyclotomic $\mathbb{Z}_p$-extension, and K_n the n-th layer. Put $\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$. We consider

$$A_{K_\infty} = \varinjlim A_{K_n}$$

which is a discrete Λ_{K_∞}-module. Recall that Δ is the maximal subgroup of G with order prime to p. We write $\mathrm{Gal}(K_\infty/k) = \Delta \times \Gamma$ where Γ is a pro-p abelian group. By the action of the complex conjugation ρ, we decompose $A_{K_\infty} = A_{K_\infty}^+ \oplus A_{K_\infty}^-$. The module we want to know is the Pontryagin dual $(A_{K_\infty}^-)^\vee$, which is a finitely generated torsion Λ_{K_∞}-module. As we saw for A_K^-, using the action of Δ, we have

$$A_{K_\infty}^- \simeq \bigoplus_{\substack{\chi \in \hat{\Delta}/\sim \\ \chi(\rho)=-1}} A_{K_\infty}^\chi .$$

We study $(A_{K_\infty}^\chi)^\vee$, which is a compact $\Lambda_{K_\infty}^\chi \simeq O_\chi[[\Gamma]]$-module.

3.1 The case $\chi \neq \omega$

We first consider the case $\chi \neq \omega$. The Stickelberger elements $\theta_{K_n/k,S}^\chi \in O_\chi[\mathrm{Gal}(K_n/k)_p]$ for $n \gg 1$ form a projective system, and define an element $\theta_{K_\infty/k,S}^\chi \in O_\chi[[\Gamma]]$ for S such that $S \supset S_{\mathrm{ram}}(K_\infty/k)$. This is essentially the p-adic L-function of Deligne and Ribet (for the relation between Stickelberger elements and the p-adic L-functions, see §4.1, especially (11)). If $S = S_{\mathrm{ram}}(K_\infty/k)$, we simply write $\theta_{K_\infty/k}^\chi$ for $\theta_{K_\infty/k,S}^\chi$.

Let S_p be the set of all p-adic places of k. We have the following simple description of the Fitting ideal of $(A^{\chi}_{K_\infty})^\vee$.

Theorem 3.1. (cf. [25] Theorem 4.4) *Suppose that χ is an odd character of Δ and $\chi \neq \omega$. Then we have*

$$\mathrm{Fitt}_{O_\chi[[\Gamma]]}((A^{\chi}_{K_\infty})^\vee) = ((\prod_{v \in S_{\mathrm{ram}}(K_\infty/k)\setminus(S_p\cup S_\infty)} (1, \frac{N_{I_v}}{1-\mathrm{Frob}_v^{-1}}))\theta^{\chi}_{K_\infty/k})^{\#}.$$

Remark 3.2. (1) In the above product, v is not a p-adic prime, so I_v is a finite group. Also, $1-\mathrm{Frob}_v^{-1}$ is a non-zero divisor in $O_\chi[[\Gamma]]$, and for any subset $J \subset S_{\mathrm{ram}}(K_\infty/k)\setminus(S_p\cup S_\infty)$, we can show that $(\prod_{v\in J}(\frac{N_{I_v}}{1-\mathrm{Frob}_v^{-1}}))\theta^{\chi}_{K_\infty/k}$ is in $O_\chi[[\Gamma]]$. This is an advantage of working over the cyclotomic $\mathbb{Z}_p$-extension because we cannot get this expression when we work over K since $1-\mathrm{Frob}_v^{-1}$ may be a zero divisor in $\mathbb{Z}_p[G]$.
(2) This theorem was proved in [23] Appendix under the assumption of the Leopoldt conjecture for k and $\mu = 0$ for K, and was (essentially) proved in [25] Theorem 4.4 under the assumption of $\mu = 0$ for K. But this assumption can be removed because Theorem 2.8 implies the above theorem, which we will explain below.

We will state and prove the T-smoothed version of Theorem 3.1.

Let T be a set of primes satisfying the conditions in Theorem 2.8. The Stickelberger elements $\theta^T_{K_\infty/k,S}$, $\theta^T_{K_\infty/k}(=\theta^T_{K_\infty/k,S_{\mathrm{ram}}(K_\infty/k)})$ in $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$ are defined similarly.

Theorem 3.3. ([25] Theorem 4.4) *We have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]^-}(((A^T_{K_\infty})^-)^\vee) = ((\prod_{v \in S_{\mathrm{ram}}(K_\infty/k)\setminus(S_p\cup S_\infty)} (1, \frac{N_{I_v}}{1-\mathrm{Frob}_v^{-1}}))\theta^{T}_{K_\infty/k})^{-,\#}.$$

Proof. The proof given in [25] is Iwasawa theoretic, but in order to explain that the vanishing of the μ-invariant is not needed, we give a proof of this theorem, which uses Theorem 2.8. Put $S = S_{\mathrm{ram}}(K_\infty/k)$. Using Theorem 2.8 and

$$\mathrm{Fitt}_{\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]^-}(((A^T_{K_\infty})^-)^\vee) = \varprojlim \mathrm{Fitt}_{\mathbb{Z}_p[\mathrm{Gal}(K_n/k)]^-}(((A^T_{K_n})^-)^\vee)$$

by Theorem 2.1 in Greither and the author [14], we have only to prove

$$\varprojlim \Theta(K_n/k)^T_p = ((\prod_{v \in S\setminus(S_p\cup S_\infty)} (1, \frac{N_{I_v}}{1-\mathrm{Frob}_v^{-1}}))\theta^{T}_{K_\infty/k}).$$

For $v \in S \setminus (S_p \cup S_\infty)$, the inertia subgroup I_v in $\mathrm{Gal}(K_\infty/k)$ is isomorphic to the inertia subgroup of v in $\mathrm{Gal}(K_n/k)$ for all n because v is unramified in K_∞/K. Let J be a subset of $S \setminus (S_p \cup S_\infty)$, and let $K_{\infty,J}$ be the fixed subfield in K_∞ of the subgroup generated by I_v for all $v \in J$. If J is empty, we define $K_{\infty,J} = K_\infty$. Since all the primes of K_n above p are totally ramified in K_∞/K_n for sufficiently large n, $\varprojlim \Theta(K_n/k)_p^T$ is generated by

$$\{N_J(\theta^T_{K_{\infty,J}/k,S\setminus J}) \mid J \subset S \setminus (S_p \cup S_\infty)\}.$$

The image of $\theta^T_{K_\infty/k}$ in $\mathbb{Z}_p[[\mathrm{Gal}(K_{\infty,J}/k)]]$ is $(\prod_{v\in J}(1-\mathrm{Frob}_v^{-1}))\theta^T_{K_{\infty,J}/k,S\setminus J}$. Therefore, noting that $1-\mathrm{Frob}_v^{-1}$ is a non-zero divisor in $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$, we have

$$N_J(\theta^T_{K_{\infty,J}/k,S\setminus J}) = (\prod_{v\in J} \frac{N_{I_v}}{1-\mathrm{Frob}_v^{-1}})\theta^T_{K_\infty/k}.$$

Thus we obtain the desired equality we first mentioned. □

We can show that Theorem 3.3 implies Theorem 3.1 by the same method as the proof of Theorem 2.4 in §2.5.

We put $\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$, and define $Q(\Lambda_{K_\infty})$ to be the total quotient ring of Λ_{K_∞}. Let γ be a topological generator of $\mathrm{Gal}(K_\infty/K)$, and $\kappa : \mathrm{Gal}(K_\infty/k) \to \mathbb{Z}_p^\times$ the cyclotomic character giving the action on μ_{p^∞}. Since $\gamma - \kappa(\gamma)$ is in $\mathrm{Ann}(\mu_{p^\infty})$, $(\gamma-\kappa(\gamma))\theta_{K_\infty/k} \in \Lambda_{K_\infty}$ can be defined as a projective system by using (4). Thus, we can define $\theta_{K_\infty/k} \in Q(\Lambda_{K_\infty})$ since $\gamma - \kappa(\gamma)$ is a non-zero divisor. As in the proof of Theorem 3.3, we put $S = S_{\mathrm{ram}}(K_\infty/k)$ and define a subfield $K_{\infty,J}$ of K_∞ as in the proof of Theorem 3.3 for any subset J of $S \setminus (S_p \cup S_\infty)$. Also, $N_J : Q(\Lambda_{K_{\infty,J}}) \to Q(\Lambda_{K_\infty})$ is defined by the multiplication by $\prod_{v\in J} N_{I_v}$. Motivated by the above proof, we define $\Theta(K_\infty/k)$ to be the Λ_{K_∞}-submodule of $Q(\Lambda_{K_\infty})$ generated by

$$\{N_J(\theta_{K_{\infty,J}/k,S\setminus J}) \mid J \subset S \setminus (S_p \cup S_\infty)\}.$$

By the integral property (4), we have

$$\mathrm{Ann}_{\Lambda_{K_\infty}}(\mu_{p^\infty}(K_\infty))\Theta(K_\infty/k) \subset \Lambda_{K_\infty}.$$

Let χ be a character of Δ such that $\chi \neq \omega$, and $\Theta(K_\infty/k)^\chi$ its χ-component in $Q(\Lambda_{K_\infty})^\chi$. Then the above integral propery implies that $\Theta(K_\infty/k)^\chi \subset \Lambda^\chi_{K_\infty} = O_\chi[[\Gamma]]$. Also, as we have seen, it coincides with # of the right hand side of Theorem 3.1. Therefore, Theorem 3.1 can be also stated as

$$\mathrm{Fitt}_{O_\chi[[\Gamma]]}((A^\chi_{K_\infty})^\vee) = (\Theta(K_\infty/k)^\chi)^\#. \tag{8}$$

3.2 The case that K/k is unramified outside p and $\chi = \omega$

Now we consider the case that $\chi = \omega$, in which our main interest is.

In general, let L/k be a Galois extension and H an abelian subgroup of order prime to p in $\mathrm{Gal}(L/k)$. For a character χ of H, we denote by L_χ the subfield of L corresponding to $\mathrm{Ker}\,\chi$. Then the natural map $A_{L_\chi}^{\chi} \to A_L^{\chi}$ is bijective by norm argument. Therefore, we may assume that $\mu_p \subset K$ and $K/k(\mu_p)$ is a p-extension. Namely, we may assume that $\Delta = \mathrm{Gal}(k(\mu_p)/k)$. So $\mathrm{Gal}(K_\infty/k) = \Delta \times \Gamma$, and $\Gamma = \mathrm{Gal}(K_\infty/k(\mu_p))$ is a pro-p group. We consider $\Lambda_{K_\infty}^{\omega} \simeq O_\omega[[\Gamma]]$ (note that $O_\omega \simeq \mathbb{Z}_p$, so $\Lambda_{K_\infty}^{\omega} \simeq \mathbb{Z}_p[[\Gamma]]$). We are interested in $A_{K_\infty}^{\omega}$ which is a $\Lambda_{K_\infty}^{\omega}$-module.

We denote by F_∞ the Δ-fixed subfield of K_∞. Then F_∞ is a totally real field, and $\mathrm{Gal}(F_\infty/k) = \Gamma$.

We may assume that $K \cap k_\infty = k$ where k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension. In fact, since our interest is in A_{K_∞} and we can always take K' such that $K_\infty = K'_\infty$ and $K' \cap k_\infty = k$, we do not lose any generality by the assumption $K \cap k_\infty = k$. Recall that $G = \mathrm{Gal}(K/k) = \Delta \times G_p$ where $\Delta = \mathrm{Gal}(k(\mu_p)/k)$ and G_p is a finite abelian p-group. By our assumption, we have

$$\Gamma = \mathrm{Gal}(K_\infty/k(\mu_p)) \simeq G_p \times \mathbb{Z}_p,$$

and $\Lambda_{K_\infty}^{\omega} = O_\omega[G_p][[\mathrm{Gal}(K_\infty/K)]]$.

We first assume in this subsection that K/k is *unramified outside* p. Then $S_{\mathrm{ram}}(K_\infty/k) \setminus (S_p \cup S_\infty)$ is the empty set, so it can be easily seen that the fractional ideal $\Theta(K_\infty/k)^\omega$ defined at the end of the previous subsection is generated by $\theta_{K_\infty/k}$, namely

$$\Theta(K_\infty/k)^\omega = \theta_{K_\infty/k}^{\omega} O_\omega[[\Gamma]]. \tag{9}$$

Suppose that

$$G_p \simeq \mathbb{Z}/p^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p^{n_s}\mathbb{Z}$$

with $0 < n_1 \le \ldots \le n_s$ for some $s \in \mathbb{Z}_{>0}$. We first define integers m_0, m_1,...,$m_{s(s-1)/2}$ as in [17] §1.2. Put

$$\varphi(\alpha) = \frac{1}{2}\alpha(2s - \alpha - 1).$$

We define $m_0 = 0$ and if v satisfies $\varphi(\alpha) < v \le \varphi(\alpha+1)$ for some integer $\alpha \in \mathbb{Z}$ with $0 \le \alpha \le s-2$, then m_v is defined by

$$m_v = (s-1)n_1 + \ldots + (s-\alpha)n_\alpha + (v - \varphi(\alpha))n_{\alpha+1}\ .$$

In particular, $m_v = vn_1$ for $0 \leq v \leq s-1$, and

$$m_{\frac{s(s-1)}{2}} = (s-1)n_1 + (s-2)n_2 + \ldots + n_{s-1} \ .$$

Let $\mathcal{A}_{K_\infty} = \mathrm{Ann}_{\Lambda^{\omega}_{K_\infty}}(\mu_{p^\infty})$ be the annihilator ideal of the group of p-power roots of unity. Note that $\mu_{p^\infty} \subset K_\infty^\times$. We define an ideal $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})$ of $\Lambda^{\omega}_{K_\infty}$ by

$$\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) = \sum_{v=0}^{\frac{s(s-1)}{2}} p^{m_v} \, \mathcal{A}_{K_\infty}^{\frac{s(s-1)}{2}-v},$$

which is determined only by $\mathcal{A}_{K_\infty}$ and the structure of G_p as an abelian group.

We note the following properties on this ideal $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})$.

(i) $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) \subset \mathfrak{m}_{\Lambda^{\omega}_{K_\infty}}^{\frac{s(s-1)}{2}}$ where $\mathfrak{m}_{\Lambda^{\omega}_{K_\infty}}$ is the maximal ideal of $\Lambda^{\omega}_{K_\infty}$.

(ii) If $n_1 = ... = n_s = m$, then $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) = (p^m, \mathcal{A}_{K_\infty})^{\frac{s(s-1)}{2}}$.

(iii) In particular, if $G_p \simeq (\mathbb{Z}/p\,\mathbb{Z})^{\oplus s}$, then $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) = \mathfrak{m}_{\Lambda^{\omega}_{K_\infty}}^{\frac{s(s-1)}{2}}$.

Theorem 3.4. (see [15] Theorem 3.3 (b) and [17] Theorem 1.2) *Assume that K/k is unramified outside p and that $K \cap k_\infty = k$. Then we have*

$$\begin{aligned} \mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}((A^{\omega}_{K_\infty})^\vee) &= (\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) \, \mathcal{A}_{K_\infty} \, \theta^{\omega}_{K_\infty/k})^\# \\ &= (\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) \, \mathcal{A}_{K_\infty} \, \Theta(K_\infty/k)^\omega)^\#. \end{aligned}$$

We prove this theorem in §5.1.

Remark 3.5. (1) By the property (4) in §2.3, we know $\mathcal{A}_{K_\infty} \theta^{\omega}_{K_\infty/k} \subset \Lambda^{\omega}_{K_\infty}$. So the right hand side of the equation in Theorem 3.4 is in $\Lambda^{\omega^{-1}}_{K_\infty}$.
(2) If $s \leq 1$, G_p is cyclic and $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) = \Lambda^{\omega}_{K_\infty}$. Theorem 3.4 says that $\mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}((A^{\omega}_{K_\infty})^\vee) = (\mathcal{A}_{K_\infty} \, \theta^{\omega}_{K_\infty/k})^\#$ in this case. On the other had, if $s \geq 2$, we have $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty}) \subset \mathfrak{m}_{\Lambda^{\omega}_{K_\infty}}$. Therefore, Theorem 3.4 implies that $(\mathcal{A}_{K_\infty} \, \theta^{\omega}_{K_\infty/k})^\#$ is not in $\mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}((A^{\omega}_{K_\infty})^\vee)$ whenever $s \geq 2$. Compare these results with Corollary 2.9 (2).
(3) Put $X = (A^{\omega}_{K_\infty})^\vee$. For a character ψ of G_p, consider the ψ-quotient $X_\psi = X \otimes_{\mathbb{Z}_p[G_p]} O_\psi$. Then its characteristic ideal $\mathrm{char}(X_\psi)$ as an $O_\psi[[\mathrm{Gal}(K_\infty/K)]]$-module does not contain information on $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})$ because $\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})$ is an ideal of finite index. Thus the above theorem gives finer information, which is not obtained by the usual main conjecture.

We define $\mathcal{A}_K \subset O_\omega[G_p]$ to be the annihilator ideal of $\mu_{p^\infty}(K)$. We define an ideal $\mathfrak{a}_{G_p}(\mathcal{A}_K)$ of $O_\omega[G_p]$ by

$$\mathfrak{a}_{G_p}(\mathcal{A}_K) = \sum_{v=0}^{\frac{s(s-1)}{2}} p^{m_v} \, \mathcal{A}_K^{\frac{s(s-1)}{2}-v}. \tag{10}$$

Corollary 3.6. *Assume that K/k is unramified outside p, $K \cap k_\infty = k$, and $\omega(\mathfrak{p}) \neq 1$ for any $\mathfrak{p} \in S_p$. Then we have*

$$\mathrm{Fitt}_{O_{\omega^{-1}}[G_p]}((A_K^\omega)^\vee) = (\mathfrak{a}_{G_p}(\mathcal{A}_K)\, \mathcal{A}_K\, \theta_{K/k}^\omega)^\#.$$

Proof. Let $c : \Lambda_{K_\infty}^\omega = O_\omega[[\Gamma]] \to O_\omega[G_p]$ be the natural restriction map. Then we have $c(\mathcal{A}_{K_\infty}) = \mathcal{A}_K$ and $c(\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})) = \mathfrak{a}_{G_p}(\mathcal{A}_K)$. By our assumption of the non-existence of trivial zeros ($\omega(\mathfrak{p}) \neq 1$ for all $\mathfrak{p} \in S_p$), the natural map $A_K^\omega \to (A_{K_\infty}^\omega)^{\mathrm{Gal}(K_\infty/K)}$ is bijective. Therefore, taking the images under the map c of both sides of the equation in Theorem 3.4, we get this corollary. □

Let us go back to the numerical example in §2.4 that $k = \mathbb{Q}(\sqrt{1901})$, $K = k(\sqrt{-3}, \alpha, \beta)$, and $p = 3$. This example satisfies all the assumptions of Corollary 3.6. In this example, $s = 2$, $\mathcal{A}_K$ is the maximal ideal $\mathfrak{m}$ of $O_\omega[G_p]$, and $\mathfrak{a}_{G_p}(\mathcal{A}_K) = \mathfrak{m}^{\frac{s(s-1)}{2}} = \mathfrak{m}$. Therefore, we theoretically recover the equality (5) in §2.4 from Corollary 3.6.

3.3 The case that K/k is not unramified outside p and $\chi = \omega$

In the previous subsection, we treated the case that K/k is unramified outside p. In this subsection, we treat the remaining case, namely we assume that there is a non p-adic prime which ramifies in K/k.

In this case, we need an important integral element

$$\vartheta_{K_\infty/k} \in \Lambda_{K_\infty},$$

whose definition we start with. We put $S = S_{\mathrm{ram}}(K_\infty/k)$ and $S' = S \setminus (S_p \cup S_\infty)$. Then, by our assumption in this subsection, S' is not empty. For a subset J of S', let N_J, $K_{\infty,J}$,...be as in §3.1. We define

$$\vartheta_{K_\infty/k} = \sum_{J \subset S'} N_J((\prod_{v \in J} \frac{N(v)^{-1} - 1}{\# I_v}\, \mathrm{Frob}_v)\theta_{K_{\infty,J}/k, S\setminus J})$$

where J runs over all subsets of S'. Here, $N(v)$ is the norm of v and Frob_v is the Frobenius of v. Since $N(v) \equiv 1 \pmod{\# I_v}$ by local class field theory, we know that $\vartheta_{K_\infty/k}$ is in $\Theta(K_\infty/k)$ where $\Theta(K_\infty/k)$ is the fractional ideal defined in the end of §3.1. An important property of $\vartheta_{K_\infty/k}$ is that it is in Λ_{K_∞} (it is a holomorphic p-adic L-function). In fact, in order to show this integrality, the problem lies only in showing the integrality of the ω-component $\vartheta^{\omega}_{K_\infty/k}$. We will prove $\vartheta^{\omega}_{K_\infty/k} \in \Lambda^{\omega}_{K_\infty}$ in §5.2 (18). (This reduces to the property $h_{F_\infty/k} \in \Lambda_{F_\infty}$ for a certain element $h_{F_\infty/k}$, which we will define in §5.2.)

This element

$$\vartheta_{K_\infty/k} \in \Theta(K_\infty/k) \cap \Lambda_{K_\infty}$$

plays the central role in describing the Fitting ideal instead of $\theta_{K_\infty/k}$. This element $\vartheta_{K_\infty/k}$ was introduced by Greither on page 753 in [12] (the notation Ψ_S was used in [12]).

Note that this element is defined only in the case $S' \neq \emptyset$ (so we cannot use this element in the previous subsection). Also, this element is totally different from the T-modification of Stickelberger elements.

We next introduce Kataoka's shifted Fitting ideals. Let R be a commutative noetherian local ring, and M a finitely generated torsion R-module. Suppose that

$$0 \to N \to P_1 \to ... \to P_r \to M \to 0$$

is an exact sequence of finitely generated torsion R-modules such that P_i is of projective dimension ≤ 1 for all i. We define the fractional ideal $\mathrm{Fitt}^{[r]}_R(M)$ of R, called r-th shifted Fitting ideal of M, by

$$\mathrm{Fitt}^{[r]}_R(M) = \prod_{i=1}^{r} \mathrm{Fitt}_R(P_i)^{(-1)^i}\,\mathrm{Fitt}_R(N).$$

Then the right hand side is independent of the choice of the exact sequence, and this is well-defined (Kataoka [20] Theorem 2.6).

Let $D_v(K_\infty/k)$ be the decomposition group of $\mathrm{Gal}(K_\infty/k)$ for $v \in S'$. This is a subgroup of finite index. We define $Z^0_{K_\infty}$ by

$$Z^0_{K_\infty} = \mathrm{Ker}(\bigoplus_{v\in S'} \mathbb{Z}_p[\mathrm{Gal}(K_\infty/k)/D_v(K_\infty/k)] \to \mathbb{Z}_p)$$

where the above map is induced by the augmentation homomorphisms of $\mathrm{Gal}(K_\infty/k)/D_v(K_\infty/k)$.

We consider $Z^0_{K_\infty}(-1)$ where (-1) is the Tate twist, and its ω^{-1}-component $(Z^0_{K_\infty}(-1))^{\omega^{-1}}$ which is a $\Lambda^{\omega^{-1}}_{K_\infty}$-module. We consider $\mathrm{Fitt}^{[1]}_{\Lambda^{\omega^{-1}}_{K_\infty}}((Z^0_{K_\infty}(-1))^{\omega^{-1}})$. Then we have

Theorem 3.7. (see [16] Theorem 0.1) *Assume that there is a non p-adic prime which ramifies in K/k. Then we have*

$$\mathrm{Fitt}_{\Lambda_{K_\infty}^{\omega^{-1}}}((A_{K_\infty}^{\omega})^\vee) = \mathrm{Fitt}^{[1]}_{\Lambda_{K_\infty}^{\omega^{-1}}}((Z^0_{K_\infty}(-1))^{\omega^{-1}})\vartheta^{\omega,\#}_{K_\infty/k}.$$

We prove this theorem in §5.2.

Remark 3.8. (1) For more explicit descriptions of $\mathrm{Fitt}^{[1]}_{\Lambda_{K_\infty}^{\omega^{-1}}}((Z^0_{K_\infty}(-1))^{\omega^{-1}})$, see [16] §§4 and 5. When $\#G_p = p$ (i.e. $[K : k(\mu_p)] = p$), an explicit description is given in [24] Theorem 0.1.
(2) If $\omega(\mathfrak{p}) \neq 1$ for all $\mathfrak{p} \in S_p$, then from Theorem 3.7 we can get the description of $\mathrm{Fitt}_{\mathcal{O}_{\omega^{-1}}[G_p]}((A_K^{\omega})^\vee)$ as in Corollary 3.6.

4 Equivariant Iwasawa Main Conjecture and its proof

In this section, we formulate and prove the Equivariant Iwasawa Main Conjecture for abelian extensions of totally real fields, including the case $p = 2$. Most papers including Ritter and Weiss [31] and Johnston and Nickel [19] assume $p > 2$, so we give here a clear view on how we can deduce the Equivariant Iwasawa Main Conjecture from Theorem 2.12 by Dasgupta, Kakde, Silliman and Wang, which we stated in §2.6, including the case $p = 2$. We also recommend the readers to see Johnston and Nickel [19].

We will use in §5 the Equivariant Iwasawa Main Conjecture for totally real fields to prove Theorems 3.4, 3.7. The relation between Iwasawa modules over totally real fields and minus class groups is given in §4.3.

From this section, p is any prime number ($p = 2$ is allowed). We consider a finite abelian extension F/k of totally real number fields and the cyclotomic $\mathbb{Z}_p$-extension F_∞/F. We put $\Lambda_{F_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/k)]]$.

Let S_∞, S_p be the sets of infinite places and of p-adic places of k, respectively. We also denote by $S_{\mathrm{ram}}(F/k)$ the set of places ramifying in F/k. In this section, we assume that S is a finite set of places of k such that

$$S \supset S_\infty \cup S_p \cup S_{\mathrm{ram}}(F/k).$$

4.1 formulation

First of all, for a number field L and a finite set S such that $S \supset S_\infty \cup S_{\mathrm{ram}}(L/k)$, we use a perfect complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ by Burns and Flach in [3] Proposition 1.20, which works well even for $p = 2$. We write $R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1))$

for the projective limit of $R\Gamma_c(\mathcal{O}_{F_n,S}, \mathbb{Z}_p(1))$ where F_n is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension F_∞/F.

We denote by $M_{F_\infty,S}/F_\infty$ the maximal abelian pro-p extension that is unramified outside S, and put $X_{F_\infty,S} = \mathrm{Gal}(M_{F_\infty,S}/F_\infty)$, which is a torsion Λ_{F_∞}-module. Since the weak Leopoldt conjecture

$$H^2_{\mathrm{et}}(\mathcal{O}_{K_\infty,S}, \mathbb{Q}_p\,/\,\mathbb{Z}_p) = H^2(\mathrm{Gal}(\mathcal{L}_S/K_\infty), \mathbb{Q}_p\,/\,\mathbb{Z}_p) = 0$$

holds where $\mathcal{L}_S/K_\infty$ is the maximal unramified extension outside S, H^i of $C^\bullet = R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1))$ is zero except for $i = 2$, 3. Using the isomorphism on Page 86 line 6 in [3], we get $H^2(C^\bullet) = X_{F_\infty,S}$ and $H^3(C^\bullet) = \mathbb{Z}_p$. Thus, we regard $C^\bullet$ as an object of $D^{\mathrm{perf}}_{\mathrm{tors}}(\Lambda_{F_\infty})$ which is the derived category of perfect complexes of Λ_{F_∞}-modules, whose cohomology groups are torsion Λ_{F_∞}-modules.

Put $K_\infty = F_\infty(\mu_{2p})$ and $\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$. We write $\Lambda^\sim_{K_\infty}$ for the module of pseudo-measures of $\mathrm{Gal}(K_\infty/k)$ in the sense of Serre [32], namely the submodule of the total quotient ring $Q(\Lambda_{K_\infty})$ consisting of elements $x \in Q(\Lambda_{K_\infty})$ that satisfy $xI_{\mathrm{Gal}(K_\infty/k)} \subset \Lambda_{K_\infty}$ where $I_{\mathrm{Gal}(K_\infty/k)}$ is the augmentation ideal of Λ_{K_∞}.

Let $\kappa : \mathrm{Gal}(K_\infty/k) \to \mathbb{Z}_p^\times$ be the cyclotomic character. For any charater ψ of finite order, we denote by $L_S(s, \psi)$ the S-truncated L-function for ψ. Then the p-adic L-function $g_{K_\infty/k,S}$ of Deligne and Ribet is the element in $\Lambda^\sim_{K_\infty}$ (see [32]) satisfying

$$\kappa^n\psi(g_{K_\infty/k,S}) = L_S(1 - n, \psi)$$

for any character ψ of $\mathrm{Gal}(K_\infty/k)$ of finite order and for any positive integer $n \in \mathbb{Z}_{>0}$, where we extend a character $\kappa^n\psi : \mathrm{Gal}(K_\infty/k) \to \overline{\mathbb{Q}}_p^\times$ to a ring homomorphism $\Lambda_{K_\infty} \to \overline{\mathbb{Q}}_p$, and also to $\Lambda^\sim_{K_\infty} \to \overline{\mathbb{Q}}_p$.

We define $g_{F_\infty/k,S} \in \Lambda^\sim_{F_\infty}$ to be the image of $g_{K_\infty/k,S}$ under the natural restriction map $\Lambda^\sim_{K_\infty} \to \Lambda^\sim_{F_\infty}$.

The relation between $g_{F_\infty/k,S}$ and Stickelberger elements in §3 is given as follows. Consider the Stickelberger element $\theta_{K_\infty/k,S} \in Q(\Lambda_{K_\infty})$ defined in §3.1 where $Q(\Lambda_{K_\infty})$ is the total quotient ring of $\Lambda_{K_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/k)]]$. We denote by

$$\tau : Q(\Lambda_{K_\infty}) \to Q(\Lambda_{K_\infty})$$

the twist automorphism induced by $\tau(\sigma) = \kappa(\sigma)^{-1}\sigma$ for all $\sigma \in \mathrm{Gal}(K_\infty/k)$. Then we have

$$\tau(\theta^{\#}_{K_\infty/k,S}) = g_{K_\infty/k,S}, \tag{11}$$

so $g_{F_\infty/k,S}$ is the image of $\tau(\theta^{\#}_{K_\infty/k,S})$ under the restriction map $Q(\Lambda_{K_\infty}) \to Q(\Lambda_{F_\infty})$.

We denote the determinant functor $D^{\mathrm{perf}}_{\mathrm{tors}}(\Lambda_{F_\infty}) \to Q(\Lambda_{F_\infty})$ by Det. Our formulation of the Equivariant Iwasawa Main Conjecture is as follows.

Theorem 4.1. *(Equivariant Iwasawa Main Conjecture) We have*

$$\mathrm{Det}(R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1)))^{-1} = g_{F_\infty/k,S}\Lambda_{F_\infty} \ .$$

We can formulate this equality by using K-groups as in [31], [19].

4.2 a proof

We give a proof of Theorem 4.1, using Theorem 2.12. Put $K = F(\mu_{2p})$, $K_\infty = F_\infty(\mu_{2p}) = F(\mu_{p^\infty})$, and $\Delta = \mathrm{Gal}(K_\infty/F_\infty)$.

Since $\mathrm{Gal}(F_\infty/k)$ is a direct product of a finite abelian group of order prime to p and a finitely generated $\mathbb{Z}_p$-module of rank 1, it is a direct product of a finite abelian group and $\mathbb{Z}_p$. Therefore, we can take a subfield $F' \subset F_\infty$ such that $F_\infty = F'k_\infty$ and $F' \cap k_\infty = k$ where k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension. Taking $F = F'$ from the first, we may assume that $\mathrm{Gal}(F_\infty/k) \simeq \mathrm{Gal}(F/k) \times \mathrm{Gal}(k_\infty/k)$. This also implies that $\mathrm{Gal}(K_\infty/k) \simeq \mathrm{Gal}(K/k) \times \mathrm{Gal}(k_\infty/k)$.

If we prove the equality in Theorem 4.1 for F_∞/k, then for any intermediate field M of F/k and its cyclotomic $\mathbb{Z}_p$-extension M_∞, we can get the equality in Theorem 4.1 for M_∞/k since $g_{M_\infty/k,S}$ is the image of $g_{F_\infty/k,S}$ under the natural restriction map. Therefore, in order to prove Theorem 4.1, we may assume that $F = K^+$, the maximal totally real subfield of K. Then K/F is a quadratic extension.

We take T as in Theorem 2.12, and consider $\mathrm{Sel}^T_S(K_n)_p$ for the n-th layer K_n of K_∞/K. One can define naturally the norm map from $\mathrm{Sel}^T_S(K_m)_p$ to $\mathrm{Sel}^T_S(K_n)_p$ for $m > n$, and $\mathrm{Sel}^T_S(K_\infty)_p$ is defined as the projective limit of $\mathrm{Sel}_S(K_n)_p$. We also define $\mathrm{Sel}_S(K_\infty)_p$, similarly. Proposition 2.4 (i) in [5] tells us that there is a distinguished triangle

$$R\Gamma_c(\mathcal{O}_{K_\infty,S}, \mathbb{Z}_p) \to R\Gamma_{c,T}(\mathcal{O}_{K_\infty,S}, \mathbb{Z}_p) \to (\bigoplus_{w \in T_{K_\infty}} \mathbb{Q}_p\,/\,\mathbb{Z}_p(1))^\vee[-2] \to \ .$$

The second cohomology of the first two complexes are $\mathrm{Sel}_S(K_\infty)_p$, $\mathrm{Sel}^T_S(K_\infty)_p$, respectively (see Proposition 2.4 (iii) in [5]), and the third cohomology of them are $(\mathbb{Q}_p\,/\,\mathbb{Z}_p(1))^\vee$, 0, respectively. It follows that we have an exact sequence

$$0 \to \mathrm{Sel}_S(K_\infty)_p \to \mathrm{Sel}_S^T(K_\infty)_p \to \bigoplus_{w \in T_{K_\infty}} \mathbb{Z}_p(-1) \to \mathbb{Z}_p(-1) \to 0. \qquad (12)$$

Recall that K/F is a quadratic extension. The Galois group $\Delta' = \mathrm{Gal}(K/F)$ is generated by the complex conjugation ρ. For any Δ'-module M, we define M_- to be $M/(1+\rho)$ as in §2.6.

The relationship between $\mathrm{Det}(R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1)))$ and Selmer modules in §2.6 is given by the following lemma.

Lemma 4.2. (1) *We have an isomorphism*

$$(\mathrm{Sel}_S(K_\infty)_p)_-(1) \simeq X_{F_\infty,S}$$

of Λ_{F_∞}*-modules for* $p > 2$ *where* (1) *is the Tate twist. For* $p = 2$*, there is a natural injective homomorphism* $(\mathrm{Sel}_S(K_\infty)_p)_-(1) \to X_{F_\infty,S}$ *whose cokernel is of order* 2*.*
(2) *We have*

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}((\mathrm{Sel}_S^T(K_\infty)_p)_-(1)) = (\prod_{v \in T}(1 - \mathrm{Frob}_v))g_{F_\infty/k,S}\Lambda_{F_\infty} \ .$$

Proof. (1) By Proposition 2.11, the Pontryagin dual of $\mathrm{Sel}_S(K_\infty)_p$ is isomorphic to $H^1_{\mathrm{et}}(\mathcal{O}_{K_\infty,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) = H^1(\mathrm{Gal}(\mathcal{L}_S/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p(1))$ where $\mathcal{L}_S/K_\infty$ is the maximal unramified extension outside S. Therefore, for $p > 2$ the Pontryagin dual of $(\mathrm{Sel}_S(K_\infty)_p)_-$ is

$$\begin{aligned} H^1(\mathrm{Gal}(\mathcal{L}_S/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p(1))^{\rho=-1} &= H^1(\mathrm{Gal}(\mathcal{L}_S/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^{\Delta'}(1) \\ &= H^1(\mathrm{Gal}(\mathcal{L}_S/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)(1) \\ &= \mathrm{Hom}(X_{F_\infty,S}, \mathbb{Q}_p/\mathbb{Z}_p)(1) \end{aligned}$$

since $H^i(\Delta', \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for $i \geq 1$. Taking the Pontryagin dual again, we obtain $(\mathrm{Sel}_S(K_\infty)_p)_- \simeq X_{F_\infty,S}(-1)$.

For $p = 2$, we know $H^1(\Delta', \mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathbb{Z}/2\mathbb{Z}$, and $H^2(\Delta', \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Therefore, the natural map from $\mathrm{Hom}(X_{F_\infty,S}, \mathbb{Q}_p/\mathbb{Z}_p) = H^1(\mathrm{Gal}(\mathcal{L}_S/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ to $H^1(\mathrm{Gal}(\mathcal{L}_S/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^{\Delta'} = ((\mathrm{Sel}_S(K_\infty)_p)^\vee)^{\rho=-1}(-1)$ is surjective and has kernel $\simeq \mathbb{Z}/2\mathbb{Z}$. Taking the Pontryagin dual, we get the conclusion.

(2) For the complex conjugation ρ, $\rho \mapsto \pm 1$ induces a ring homomorphism $\Lambda_{K_\infty} \to \Lambda_{F_\infty}$, which we denote by $x \mapsto x_\pm$, respectively. By Theorem 2.12 we have

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(\mathrm{Sel}_S^T(K_\infty)_p)_- = ((\theta^T_{K_\infty/k,S})^\#)_-\Lambda_{F_\infty} \ .$$

We defined $\tau : \Lambda_{K_\infty} \to \Lambda_{K_\infty}$ in §4.1. We define $\tau_{F_\infty} : \Lambda_{F_\infty} \to \Lambda_{F_\infty}$ to be the map induced by $\sigma \mapsto \kappa(\sigma)^{-1}\sigma$ for any $\sigma \in \mathrm{Gal}(F_\infty/k)$. Then it follows from Theorem 2.12 and (11) that we have

$$\begin{aligned}
\mathrm{Fitt}_{\Lambda_{F_\infty}}(\mathrm{Sel}_S^T(K_\infty)_p)_-(1)) &= \tau_{F_\infty}((\theta^T_{K_\infty/k,S})^\#)_-\Lambda_{F_\infty} \\
&= (\tau((\theta^T_{K_\infty/k,S})^\#))_+\Lambda_{F_\infty} \\
&= \tau((\prod_{v\in T}(1-\mathrm{Frob}_v\, N(v)))\theta^\#_{K_\infty/k,S})_+\Lambda_{F_\infty} \\
&= (\prod_{v\in T}(1-\mathrm{Frob}_v))(g_{K_\infty/k,S})_+\Lambda_{F_\infty} \\
&= (\prod_{v\in T}(1-\mathrm{Frob}_v))g_{F_\infty/k,S}\Lambda_{F_\infty}.
\end{aligned}$$

□

Now we prove Theorem 4.1. Since Theorem 4.1 is an equality of two invertible Λ_{F_∞}-modules, by [4] Lemma 6.1 we have only to prove

$$\mathrm{Det}(R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1)))_{\mathfrak{p}}^{-1} = g_{F_\infty/k,S}\Lambda_{F_\infty,\mathfrak{p}} \tag{13}$$

for every height-one prime $\mathfrak{p}$ of Λ_{F_∞}.

We take a height-one prime $\mathfrak{p}$ of Λ_{F_∞}. Then we have $M_{\mathfrak{p}} = 0$ for any finite Λ_{F_∞}-module M. Therefore, the exact sequence (12) and Lemma 4.2 (1) yield an exact sequence

$$0 \to (X_{F_\infty,S})_{\mathfrak{p}} \to (\mathrm{Sel}_S^T(K_\infty)_p)_-(1)_{\mathfrak{p}} \to (\bigoplus_{w\in T_{K_\infty}} \mathbb{Z}_p)_{\mathfrak{p}} \to (\mathbb{Z}_p)_{\mathfrak{p}} \to 0 \tag{14}$$

of $\Lambda_{F_\infty,\mathfrak{p}}$-modules. Using Lemma 4.2 (2), we have

$$\begin{aligned}
\mathrm{Det}(R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1)))_{\mathfrak{p}}^{-1} &= \mathrm{Det}_{\Lambda_{F_\infty,\mathfrak{p}}}((X_{F_\infty,S})_{\mathfrak{p}})^{-1}\,\mathrm{Det}_{\Lambda_{F_\infty,\mathfrak{p}}}((\mathbb{Z}_p)_{\mathfrak{p}}) \\
&= \mathrm{Det}_{\Lambda_{F_\infty,\mathfrak{p}}}((\mathrm{Sel}_S^T(K_\infty)_p)_-(1)_{\mathfrak{p}})^{-1}\,\mathrm{Det}_{\Lambda_{F_\infty,\mathfrak{p}}}((\bigoplus_{w\in T_{K_\infty}} \mathbb{Z}_p)_{\mathfrak{p}}) \\
&= \mathrm{Fitt}_{\Lambda_{F_\infty,\mathfrak{p}}}((\mathrm{Sel}_S^T(K_\infty)_p)_-(1)_{\mathfrak{p}})\,\mathrm{Fitt}_{\Lambda_{F_\infty,\mathfrak{p}}}((\bigoplus_{w\in T_{K_\infty}} \mathbb{Z}_p)_{\mathfrak{p}})^{-1} \\
&= g_{F_\infty/k,S}\Lambda_{F_\infty,\mathfrak{p}}\ .
\end{aligned}$$

In the above computation we used $\mathrm{Det}_R(M)^{-1} = \mathrm{Fitt}_R(M)$ for a finitely generated torsion R-module M of projective dimension ≤ 1. This completes the proof of Theorem 4.1.

We give three remarks here. At first, let $I_{\mathrm{Gal}(F_\infty/k)}$ be the augmentation ideal $\mathrm{Ker}(\Lambda_{F_\infty} = \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/k)]] \to \mathbb{Z}_p)$. If $\mathfrak{p} \neq I_{\mathrm{Gal}(F_\infty/k)}$, then $(\mathbb{Z}_p)_\mathfrak{p} = 0$, and $\mathrm{Fitt}_{\Lambda_{F_\infty,\mathfrak{p}}}(X_{F_\infty,S})_\mathfrak{p} = g_{F_\infty/k,S}\Lambda_{F_\infty,\mathfrak{p}}$. If $\mathfrak{p} = I_{\mathrm{Gal}(F_\infty/k)}$, then $(\mathbb{Z}_p)_\mathfrak{p} = \mathbb{Q}_p$, and we get $\mathrm{Fitt}_{\Lambda_{F_\infty,\mathfrak{p}}}(X_{F_\infty,S})_\mathfrak{p} = (\gamma-1)g_{F_\infty/k,S}\Lambda_{F_\infty,\mathfrak{p}}$ where γ is a topological generator of $\mathrm{Gal}(k_\infty/k)$.

Secondly, using the same method, we can easily obtain the usual Iwasawa main conjecture, proved by Wiles [36]. For a character ψ of $\mathrm{Gal}(F/k)$, we write $O_\psi = \mathbb{Z}_p[\mathrm{Image}\ \psi]$ on which $\mathrm{Gal}(F/k)$ acts via ψ, and consider the ψ-quotient $(X_{F_\infty,S})_\psi = X_{F_\infty,S} \otimes_{\mathbb{Z}_p[\mathrm{Gal}(F/k)]} O_\psi$. Then, by the same method as the above proof, we can easily see that the characteristic ideal $\mathrm{char}(X_{F_\infty,S})_\psi$ is generated by the image of $g_{F_\infty/k,S}$ in $O_\psi[[\mathrm{Gal}(k_\infty/k)]]$ if ψ is not the trivial character, and by $(\gamma-1)$ times the image of $g_{F_\infty/k,S}$ if ψ is the trivial character.

Thirdly, if p does not belong to $\mathfrak{p}$, then $\Lambda_{F_\infty,\mathfrak{p}}$ is a discrete valuation ring, and it is easy to work over $\Lambda_{F_\infty,\mathfrak{p}}$ (cf. [6] §3C1, for example). If p belongs to $\mathfrak{p}$, then we have $M_\mathfrak{p} = 0$ for any Λ_{F_∞}-module M that is finitely generated over $\mathbb{Z}_p$. Therefore, $(X_{F_\infty,S})_\mathfrak{p} \stackrel{\sim}{\to} ((\mathrm{Sel}_S^T(K_\infty)_p)_-)_\mathfrak{p}(1)$ is an isomorphism, and (13) is also obtained immediately from this isomorphism and Lemma 4.2 (2).

Corollary 4.3. *There exists an exact sequence of finitely generated torsion Λ_{F_∞}-modules*

$$0 \to X_{F_\infty,S} \to P_1 \to P_2 \to \mathbb{Z}_p \to 0$$

such that the projective dimensions of P_1, P_2 are ≤ 1, and

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(P_1)\,\mathrm{Fitt}_{\Lambda_{F_\infty}}(P_2)^{-1} = g_{F_\infty/k,S}\Lambda_{F_\infty}$$

in $Q(\Lambda_{F_\infty})$.

Proof. This seems standard (see [21] Theorem 4.1), but we give here a sketch of the proof for the convenience of the readers. We can take a complex $C^1 \stackrel{d^1}{\to} C^2 \stackrel{d^2}{\to} C^3$, which is concentrated on degrees 1, 2, 3, and which is quasi-isomorphic to $R\Gamma_c(\mathcal{O}_{F_\infty,S}, \mathbb{Z}_p(1))$ such that C^i are finitely generated and projective. We have an exact sequence

$$0 \to X_{F_\infty,S} \to C^2/d^1(C^1) \stackrel{\overline{d^2}}{\to} C^3 \to \mathbb{Z}_p \to 0.$$

Put $M = \mathrm{Ker}(C^3 \to \mathbb{Z}_p)$. Take, for example, $t = \gamma-1$ where γ is a generator of $\mathrm{Gal}(F_\infty/F)$, and consider the multiplication by t on C^3, which is injective.

Then tC^3 is in M because γ acts on $\mathbb{Z}_p$ trivially. Since $\overline{d^2} : C^2/d^1(C^1) \to M$ is surjective and C^3 is projective, there is a Λ_{F_∞}-homomorphism $f : C^3 \to C^2/d^1(C^1)$ such that $\overline{d^2} \circ f$ is the multiplication by t on C^3. Then we can construct an exact sequence

$$0 \to X_{F_\infty,S} \to \mathrm{Coker}\, f \to C^3/tC^3 \to \mathbb{Z}_p \to 0$$

where both $\mathrm{Coker}\, f$ and C^3/tC^3 are finitely generated torsion Λ_{F_∞}-modules, whose projective dimensions are ≤ 1. We can take $P_1 = \mathrm{Coker}\, f$ and $P_2 = C^3/tC^3$. Now, Theorem 4.1 implies the equation in Corollary 4.3. □

We remark that we can also construct an exact sequence in Corollary 4.3 concretely, using a variant of the method to construct the sequence (12).

4.3 Relation with minus class groups

We go back to the setting in §§3.2, 3.3, namely p is odd, and K/k is a CM-extension such that $K = F(\mu_p)$ and F/k is a finite abelian p-extension of totally real number fields. We studied the class group A_{K_∞} in §3. In this subsection we give a relationship between such minus class groups and $X_{F_\infty,S}$ which we studied in the previous subsection.

Proposition 4.4. *Suppose that p is odd. Then we have an isomorphism*

$$X_{F_\infty,S_p\cup S_\infty} \simeq (A^{\omega}_{K_\infty})^{\vee}(1).$$

Proof. We consider $\mathrm{Sel}_{S_p\cup S_\infty}(K_\infty)_p$. By Proposition 2.11 we know that the Pontryagin dual of $\mathrm{Sel}_{S_p\cup S_\infty}(K_\infty)_p$ is isomorphic to $H^1(\mathrm{Gal}(\mathcal{L}_{S_p\cup S_\infty}/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p(1))$.

Let E_{K_n} be the unit group of K_n and $E_{K_\infty} = \bigcup E_{K_n}$. By Kummer duality (see Iwasawa [18] the last line on Page 275), we have an exact sequence

$$0 \to E_{K_\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(\mathrm{Gal}(\mathcal{L}_{S_p\cup S_\infty}/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p(1)) \to A_{K_\infty} \to 0.$$

Since p is odd, $\Delta = \mathrm{Gal}(K/F)$ is of order prime to p. Thus taking the Δ-invariant parts is an exact functor. We have $(E_{K_\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p(-1))^{\Delta} = 0$ and

$$H^1(\mathrm{Gal}(\mathcal{L}_{S_p\cup S_\infty}/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^{\Delta} \simeq H^1(\mathrm{Gal}(\mathcal{L}_{S_p\cup S_\infty}/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) \simeq (X_{F_\infty,S_p\cup S_\infty})^{\vee}.$$

Therefore, taking (-1)-twist and Δ-invarint parts of the above exact sequence, we have an isomorphism

$$(X_{F_\infty,S_p\cup S_\infty})^{\vee} \simeq A^{\omega}_{K_\infty}(-1).$$

This completes the proof. □

5 S-ramified Iwasawa modules over totally real number fields

In §4.3 we saw that we have only to know $X_{F_\infty,S_p\cup S_\infty}$ to understand the ω-component of A_{K_∞}. In this section, p is any prime number, and we assume that F/k is a finite abelian p-extension of totally real number fields and denote its cyclotomic $\mathbb{Z}_p$-extension by F_∞ (the field F_∞ defined in §3.2 satisfies the above conditions). Without loss of generality, we may assume that $F \cap k_\infty = k$ where k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension. We put $G_p = \mathrm{Gal}(F/k)$ and $\Gamma = \mathrm{Gal}(F_\infty/k)$, so

$$\Gamma = \mathrm{Gal}(F_\infty/k) = G_p \times \mathrm{Gal}(k_\infty/k) \simeq G_p \times \mathbb{Z}_p \ .$$

5.1 S-ramified Iwasawa modules

In this subsection, we still assume that S is a finite set of places of k such that $S \supset S_\infty \cup S_p \cup S_{\mathrm{ram}}(F/k)$.

What we study in this subsection is the maximal abelian pro-p extension $M_{F_\infty,S}/F_\infty$ that is unramified outside S, and its Galois group $X_{F_\infty,S}$ on which $\Gamma = \mathrm{Gal}(F_\infty/k)$ acts. Put $\Lambda_{F_\infty} = \mathbb{Z}_p[[\Gamma]]$. We study $X_{F_\infty,S}$ as a Λ_{F_∞}-module.

Let $g_{F_\infty/k,S} \in \Lambda^{\sim}_{F_\infty}$ be Deligne-Ribet's p-adic L-function defined in §4.1.

We define integers $(m_v)_{v=0,1,\ldots,s(s-1)/2}$ as in §3.2. For the augmentation ideal I_Γ of Λ_{F_∞}, we define an ideal $\mathfrak{a}_{G_p}(I_\Gamma)$ of Λ_{F_∞} by

$$\mathfrak{a}_{G_p}(I_\Gamma) = \sum_{v=0}^{\frac{s(s-1)}{2}} p^{m_v} I_\Gamma^{\frac{s(s-1)}{2}-v} \tag{15}$$

which is an ideal of Λ_{F_∞} (cf. (10)).

Greither and the author proved the following theorem in Theorem 3.3 (a) in [15], and Greither, Tokio and the author gave the explicit description (15) of $\mathfrak{a}_{G_p}(I_\Gamma)$ in [17] §1.2.

Theorem 5.1. ([17] Theorem 1.2) *Assume that F/k is a finite abelian p-extension such that $F \cap k_\infty = k$ with Galois group G_p. Then we have*

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) = \mathfrak{a}_{G_p}(I_\Gamma) I_\Gamma g_{F_\infty/k,S} \ .$$

Proof. We explain here a simple proof of this theorem, using the idea of Kataoka [20]. Applying the definition of Kataoka's second shifted Fitting ideal which we explained in §3.3 to the exact sequence in Corollary 4.3, we have

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) = \mathrm{Fitt}_{\Lambda_{F_\infty}}(P_1)\,\mathrm{Fitt}_{\Lambda_{F_\infty}}(P_2)^{-1}\,\mathrm{Fitt}^{[2]}_{\Lambda_{F_\infty}}(\mathbb{Z}_p).$$

By Corollary 4.3, we get $\mathrm{Fitt}_{\Lambda_{F_\infty}}(P_1)\,\mathrm{Fitt}_{\Lambda_{F_\infty}}(P_2)^{-1} = g_{F_\infty/k,S}\Lambda_{F_\infty}$. We computed $\mathrm{Fitt}^{[2]}_{\Lambda_{F_\infty}}(\mathbb{Z}_p)$ in [17] Theorem 1.1 (see [17] Proposition 3.1) to get $\mathrm{Fitt}^{[2]}_{\Lambda_{F_\infty}}(\mathbb{Z}_p) = \mathfrak{a}_{G_p}(I_\Gamma)I_\Gamma$. These computations yield the equation in Theorem 5.1. □

Remark 5.2. (1) In [17] Theorem 1.2, the vanishing of the μ-invariant of k_∞/k and $p > 2$ are assumed. However, neither assumption is necessary as we explained in the above proof.
(2) In the above theorem, we consider the case $S \supset S_\infty$, but if we consider the Iwasawa module $X_{F_\infty,S}$ such that $S \not\supset S_\infty$, we may have to replace $g_{F_\infty/k,S}$ by $2^{-d}g_{F_\infty/k,S}$ with some d when $p = 2$.
(3) There was a guess that $\mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) = I_\Gamma g_{F_\infty/k,S}$ at some stage. Now we know by the above theorem that this guess is true only when $s \leq 1$.

Proof of Theorem 3.4. Now we prove Theorem 3.4. Recall that p is odd, $K = F(\mu_p)$ and $\Delta = \mathrm{Gal}(K/F)$. We decompose $Q(\Lambda_{K_\infty}) = \bigoplus_{\chi\in\hat{\Delta}} Q(\Lambda_{K_\infty})^\chi$ by the action of $\Delta = \mathrm{Gal}(F(\mu_p)/F)$. The χ-component $Q(\Lambda_{K_\infty})^\chi$ is $Q(\Lambda^\chi_{K_\infty}) = Q(O_\chi[[\mathrm{Gal}(F_\infty/k)]])$. For $x \in Q(\Lambda_{K_\infty})$, the χ-component of x is denoted by $x^\chi \in Q(\Lambda_{K_\infty})^\chi$.

We use the automorphism τ of $Q(\Lambda_{K_\infty})$ defined in §4.1. It is easy to check that τ gives a bijective from the ω^{-1}-component $Q(\Lambda_{K_\infty})^{\omega^{-1}}$ to the trivial character component $Q(\Lambda_{K_\infty})^{\mathbf{1}}$. We also note that the trivial character component $Q(\Lambda_{K_\infty})^{\mathbf{1}}$ coincides with $Q(\Lambda_{K_\infty})^\Delta = Q(\Lambda_{F_\infty})$. Also, $\#$ gives a bijective from $Q(\Lambda_{K_\infty})^\omega$ to $Q(\Lambda_{K_\infty})^{\omega^{-1}}$. Consider the Stickelberger element $\theta_{K_\infty/k,S} \in Q(\Lambda_{K_\infty})$ and $\theta^\#_{K_\infty/k,S} \in Q(\Lambda_{K_\infty})$. We have

$$\tau(\theta^{\omega,\#}_{K_\infty/k,S})^{\mathbf{1}} = g_{F_\infty/k,S}. \tag{16}$$

from the equation (11).

For any finitely generated torsion $\Lambda^{\omega^{-1}}_{K_\infty}$-module M, we regard $M(1)$ as a $Q(\Lambda_{K_\infty})^{\mathbf{1}} = Q(\Lambda_{F_\infty})$-module. Then we have

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(M(1)) = \tau(\mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}(M))^{\mathbf{1}}.$$

Therefore, Proposition 4.4 implies that

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,p}) = \tau(\mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}((A^\omega_{K_\infty})^\vee))^{\mathbf{1}}.$$

On the other hand, Theorem 5.1, the equality (16), $\tau(\mathcal{A}_{K_\infty}^{\#})^{\mathbf{1}} = I_\Gamma$ and $\tau(\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})^{\#})^{\mathbf{1}} = \mathfrak{a}_{G_p}(I_\Gamma)$ imply that

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,p}) = \tau((\mathfrak{a}_{G_p}(\mathcal{A}_{K_\infty})\,\mathcal{A}_{K_\infty}\,\theta^{\omega}_{K_\infty/k})^{\#})^{\mathbf{1}}.$$

Since τ is bijective, the above two equations give the desired result.

5.2 p-ramified Iwasawa modules

Next, we consider p-ramified Iwasawa modules. Namely, we consider the maximal abelian pro-p extension $M_{F_\infty,S_p\cup S_\infty}/F$ that is unramified outside p, and its Galois group $X_{F_\infty,S_p\cup S_\infty}$. We simply write $X_{F_\infty,p}$ for $X_{F_\infty,S_p\cup S_\infty}$.

If $S_{\mathrm{ram}}(F/k)\subset S_p$, then the Fitting ideal of $X_{F_\infty,p}$ is known by Theorem 5.1. So we assume that $S_{\mathrm{ram}}(F/k)\not\subset S_p$ in this subsection. We put $S = S_{\mathrm{ram}}(F_\infty/k)$ and $S' = S\setminus S_p$. By our assumption, $S'\neq\emptyset$.

We denote by $\kappa:\Gamma\to\mathbb{Z}_p^\times$ the restriction of the cyclotomic character to Γ. Then Greither, Kataoka and the author proved in [16] Theorem 1.5 the existence of an element $h_{F_\infty/k}\in\Lambda_{F_\infty}$ satisfying

$$\kappa^n\psi(h_{F_\infty/k}) = L_S(1-n,\psi\omega^{-n})\prod_{v\in S'}\frac{1-\psi(v)N(v)^n}{1-\psi(v)N(v)^{n-1}},$$

for any $n\in\mathbb{Z}_{>0}$ and any character ψ of Γ of finite order. In Lemma 2.1 in [16] we proved

$$h_{F_\infty/k} = \sum_{J\subset S'} N_J((\prod_{v\in J}\frac{N(v)^{-1}-1}{\# I_v}\,\mathrm{Frob}_v)g_{F_{\infty,J}/k,S\setminus J})$$

where N_J, Frob_v,...are as in §3.3. By (16), the above formula, and the definition of $\vartheta_{K_\infty/k}$, we have

$$\tau(\vartheta^{\omega,\#}_{K_\infty/k})^{\mathbf{1}} = h_{F_\infty/k}\ . \qquad (17)$$

Since τ gives a bijective from $\Lambda_{K_\infty}^{\omega^{-1}}$ to $\Lambda_{K_\infty}^{\mathbf{1}} = \Lambda_{F_\infty}$, the above equality together with $h_{F_\infty/k}\in\Lambda_{F_\infty}$ implies that $\vartheta^{\omega,\#}_{K_\infty/k}\in\Lambda_{K_\infty}^{\omega^{-1}}$ and

$$\vartheta^{\omega}_{K_\infty/k}\in\Lambda^{\omega}_{K_\infty}. \qquad (18)$$

We defined $Z^0_{K_\infty}$ in §3.3. Similarly, we define

$$Z^0_{F_\infty} = \mathrm{Ker}(\bigoplus_{v\in S'}\mathbb{Z}_p[\mathrm{Gal}(F_\infty/k)/D_v(F_\infty/k)]\to\mathbb{Z}_p).$$

Clearly, this is the $\Delta = \mathrm{Gal}(K/F)$-invariant part of $Z^0_{K_\infty}$, namely $(Z^0_{K_\infty})^\Delta = Z^0_{F_\infty}$. Using the notion of Kataoka's shifted Fitting ideals explained in §3.3, we consider $\mathrm{Fitt}^{[1]}_{\Lambda_{F_\infty}}(Z^0_{F_\infty})$, which is a fractional ideal determined only by decomposition groups of v for $v \in S'$ in $\Gamma = \mathrm{Gal}(F_\infty/k)$.

Then Greither, Kataoka, and the author proved the following in [16].

Theorem 5.3. ([16] Theorem 0.1) *Assume that there is a non p-adic prime which ramifies in K/k. Then we have*

$$\mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,p}) = \mathrm{Fitt}^{[1]}_{\Lambda_{F_\infty}}(Z^0_{F_\infty})h_{F_\infty/k}\ .$$

Remark 5.4. In [16], $p > 2$ is assumed. But this assumption was used only when we applied the equivariant Iwasawa main conjecture in [16] Theorem 3.11, so we can remove this assumption.

Proof of Theorem 3.7. We can prove Theorem 3.7 by the same method as the proof of Theorem 3.4 in the previous subsection. In fact, since $Z^0_{K_\infty}(-1)^{\omega^{-1}} = (Z^0_{K_\infty})^\Delta(-1) = Z^0_{F_\infty}(-1)$, we have $Z^0_{F_\infty} = Z^0_{K_\infty}(-1)^{\omega^{-1}}(1)$ and

$$\tau(\mathrm{Fitt}^{[1]}_{\Lambda^{\omega^{-1}}_{K_\infty}}(Z^0_{K_\infty}(-1)^{\omega^{-1}})) = \mathrm{Fitt}^{[1]}_{\Lambda_{F_\infty}}(Z^0_{F_\infty}).$$

Therefore, using Proposition 4.4 and (17), we get

$$\tau(\mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}((A^\omega_{K_\infty})^\vee))^{\mathbf{1}} = \mathrm{Fitt}_{\Lambda_{F_\infty}}(X_{F_\infty,p}),$$

and

$$\tau(\mathrm{Fitt}^{[1]}_{\Lambda^{\omega^{-1}}_{K_\infty}}(Z^0_{K_\infty}(-1)^{\omega^{-1}})\vartheta^{\omega,\#}_{K_\infty/k})^{\mathbf{1}} = \mathrm{Fitt}^{[1]}_{\Lambda_{F_\infty}}(Z^0_{F_\infty})h_{F_\infty/k}\ .$$

It follows from Theorem 5.3 and the bijectivity of τ that

$$\mathrm{Fitt}_{\Lambda^{\omega^{-1}}_{K_\infty}}((A^\omega_{K_\infty})^\vee) = \mathrm{Fitt}^{[1]}_{\Lambda^{\omega^{-1}}_{K_\infty}}(Z^0_{K_\infty}(-1)^{\omega^{-1}})\vartheta^{\omega,\#}_{K_\infty/k}\ .$$

This completes the proof of Theorem 3.7.

6 Generators and relations of S-ramified Iwasawa modules

This section is an exposition of the paper [21]. We consider a slightly more general setting than that in §5.1. Suppose that F/k is a finite p-extension of totally real number fields with Galois group G_p, which is not necessarily abelian. Our interest is in Iwasawa modules over the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F.

We assume that $F \cap k_\infty = k$, and put $\Gamma = \mathrm{Gal}(F_\infty/k)$. Our assumption implies that $\Gamma \simeq G_p \times \mathbb{Z}_p$. Let S be a finite set of places of k, satisfying $S \supset S_\infty \cup S_p \cup S_{\mathrm{ram}}(F/k)$. Let $M_{F_\infty,S}/F_\infty$ be the maximal abelian pro-p extension that is unramified outside S, and put $X_{F_\infty,S} = \mathrm{Gal}(M_{F_\infty,S}/F_\infty)$.

When G_p is abelian, the complicated shape of the Fitting ideal of $X_{F_\infty,S}$ (Theorem 5.1) suggests that $X_{F_\infty,S}$ is a complicated Λ_{F_∞}-module. In order to understand such complicatedness, we are interested in the minimal numbers of generators and of relations of $X_{F_\infty,S}$ as a Λ_{F_∞}-module.

For any R-module M, we denote by $\mathrm{gen}_R(M)$ the minimal number of generators of M as an R-module. Also, we define $r_R(M)$ to be the minimal number of relations of M as an R-module.

Let $M_{k,S}/k$ be the maximal abelian pro-p extension that is unramified outside S. We put

$$t = \dim_{\mathbb{F}_p} \mathrm{Gal}(M_{k,S}/M_{k,S} \cap K_\infty) \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathrm{gen}_{\mathbb{Z}_p} \mathrm{Gal}(M_{k,S}/M_{k,S} \cap K_\infty),$$

and

$$s_2 = \dim_{\mathbb{F}_p} H_2(G_p, \mathbb{F}_p) \ \text{ and } \ s_3 = \dim_{\mathbb{F}_p} H_3(G_p, \mathbb{F}_p).$$

Using the existence of the exact sequence in Corollary 4.3, Kataoka and the author proved in [21] the following.

Theorem 6.1. ([21] Theorem 3.3)

(1) $\max\{s_2, t\} \le \mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) \le s_2 + t$

(2) $r_{\Lambda_{F_\infty}}(X_{F_\infty,S}) = \mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) + s_3$

Suppose that G_p is abelian and $s = \mathrm{gen}_{\mathbb{Z}}(G_p)$ as in Theorem 5.1 where we assume $G_p = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p^{n_s}\mathbb{Z}$. Then we have $H_2(G_p, \mathbb{Z}) = \bigwedge^2 G_p$ and $H_1(G_p, \mathbb{Z}) = G_p$. These computations together with the universal coefficient theorem imply that

$$s_2 = \dim_{\mathbb{F}_p} H_2(G_p, \mathbb{F}_p) = \frac{s(s-1)}{2} + s = \frac{s(s+1)}{2}.$$

Also, we know that $s_3 = s(s+1)(s+2)/6$. In this case, K_∞ is in $M_{k,S}$. Therefore, we get

$$t = \mathrm{gen}_{\mathbb{Z}_p} \mathrm{Gal}(M_{k,S}/F_\infty).$$

From Theorem 6.1 we obtain

Corollary 6.2. ([21] Theorem 1.1) *Under the assumption of Theorem 5.1, namely the assumption that* G_p *is abelian, put* $t = \mathrm{gen}_{\mathbb{Z}_p} \mathrm{Gal}(M_{k,S}/F_\infty)$*. Then we have*

$$\max\{\frac{s(s+1)}{2}, t\} \leq \mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) \leq \frac{s(s+1)}{2} + t$$

and

$$r_{\Lambda_{F_\infty}}(X_{F_\infty,S}) = \mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) + \frac{s(s+1)(s+2)}{6}.$$

This corollary says that $\mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S})$ has quadratic growth in s. In this way, $X_{F_\infty,S}$ is surely a more complicated module than we first expected.

For numerical examples, see [21] §7. For example, take $k = \mathbb{Q}$, $F = \mathbb{Q}(\sqrt{73}, \sqrt{89}, \sqrt{97})$, $p = 2$, and $S = \{\infty, 73, 89, 97\}$. In this case, we know $s = 3$ and $t = 4$. Therefore, Corollary 6.2 says that $6 \leq \mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) \leq 10$. By numerical computation, we could check that $\mathrm{gen}_{\Lambda_{F_\infty}}(X_{F_\infty,S}) = 10$.

Finally, we go back to the setting of Corollary 3.6.

Corollary 6.3. *Assume that* K/k *is a finite abelian CM-extension satisfying all the assumptions of Corollary 3.6. We put* $F = K^\Delta$, $G_p = \mathrm{Gal}(F/k)$, $s = \mathrm{gen}_{\mathbb{Z}_p}(G_p)$*, and* $t = \mathrm{gen}_{\mathbb{Z}_p} \mathrm{Gal}(M_{k,S_p \cup S_\infty}/F_\infty)$*. Then we have*

$$\max\{\frac{s(s+1)}{2}, t\} \leq \mathrm{gen}_{\mathcal{O}_\omega[G_p]}((A_K^\omega)^\vee) \leq \frac{s(s+1)}{2} + t,$$

and

$$r_{\mathcal{O}_\omega[G_p]}((A_K^\omega)^\vee) \leq \mathrm{gen}_{\mathcal{O}_\omega[G_p]}((A_K^\omega)^\vee) + \frac{s(s+1)(s+2)}{6}$$

Proof. This follows from the isomorphisms

$$(A_K^\omega)^\vee \xleftarrow{\simeq} ((A_{K_\infty}^\omega)^\vee)_{\mathrm{Gal}(K_\infty/K)} \simeq (X_{F_\infty,S_p \cup S_\infty})(-1)_{\mathrm{Gal}(K_\infty/K)}\ .$$

□

We revisit the example in §2.4 that $k = \mathbb{Q}(\sqrt{1901})$, $K = k(\sqrt{-3}, \alpha, \beta)$, and $p = 3$. Suppose that ξ is a generator of $\mathrm{Gal}(K/F(\sqrt{-3}, \beta))$, ν is a generator of $\mathrm{Gal}(K/F(\sqrt{-3}, \alpha))$, and put $X = \xi - 1$, $Y = \nu - 1$.

Then by the numerical computation in [27] §2, we know that $(A_K^\omega)^\vee$ has 3 generators e_1, e_2, e_3 and 7 relations; $9e_1 + (XY^2 - X^2Y)e_3 = 0$, $Xe_1 - Y^2e_3 = 0$, $Ye_1 - X^2e_3 = 0$, $3e_2 + X^2Ye_3 = 0$, $Xe_2 + Y^2e_3 = 0$, $Ye_2 - X^2e_3 = 0$ and $3e_3 = 0$. (In [27] §2, one more relation $X^2Y^2e_3 = 0$ is written, but this relation is a consequence of the 4-th, 6-th, and 7-th relations.) In this example, we have $\mathrm{gen}_{\mathcal{O}_\omega[G_p]}((A_K^\omega)^\vee) = 3$ and $r_{\mathcal{O}_\omega[G_p]}((A_K^\omega)^\vee) = 7$. Thus, since $s = 2$, we could check numerically that Corollary 6.3 holds for this example.

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