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On Stark elements of arbitrary weight and their p-adic families

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Abstract.

In this article we shall generalize the theory of Stark elements to arbitrary 'weight'. This means that we define canonical generalized Stark elements that correspond to the leading terms at *arbitrary integers* of L-series attached to the multiplicative group. We investigate their basic properties, and formulate both a conjecture on the precise integrality condition that these Stark elements satisfy and a conjecture that describes the Galois module structures of certain cohomology groups as a generalization of the refined Rubin-Stark conjecture we formulated in an earlier article 'On zeta elements for $\mathbb{G}_m$' in Doc. Math. **21** (2016). We then prove these conjectures in several interesting cases. The integrality condition studied here will then play an important role in a subsequent article of the first and third authors which formulates a precise conjectural congruence relation between the generalized Stark elements of differing weights that are defined here.

§1. Introduction

1.1. Background and main results

1.1.1. The conjecture of Stark predicts that canonical elements constructed (unconditionally) from the leading terms at $s = 0$ of the Artin L-series of complex linear characters should belong to the rational vector spaces that are spanned by the r-th exterior powers of suitable groups of algebraic units, where r denotes the order of vanishing at zero of the relevant L-series.

These Stark elements satisfy some integrality properties which were first studied by Stark himself in [18], and then by Tate in [19], for the

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case $r = 1$ and then subsequently by Rubin in [15] where the so-called '*Rubin-Stark Conjecture*' was formulated in the setting of general order of vanishing.

More recently, we (the three authors of the current article) formulated in [5, Conj. 7.3] a refinement of the Rubin-Stark Conjecture that describes remarkable properties of these integral Rubin-Stark elements concerning the Galois module structures of natural Selmer modules that are closely related to ideal class groups.

At this point it would be worth remarking that a detailed study of the integrality properties of Rubin-Stark elements is of interest since the collection of such elements defined over a general number field is conjectured to constitute a higher rank Euler system for $\mathbb{G}_m$ over that field.

It seems reasonable to believe, therefore, that the detailed study of Rubin-Stark elements should give one a better understanding of any general theory of higher rank Euler systems that could be developed.

1.1.2. Our aim in the present article is now to extend the theory developed in [5] by defining 'generalized Stark elements' of 'arbitrary weight', that is, to define elements that correspond to the leading terms of L-series $L(\chi, s)$ at an arbitrary integer $s = j$, and to use the approach of [5] to investigate the integrality properties of these elements.

To be a little more precise, we fix a finite abelian extension L/K of number fields and set $G := \mathrm{Gal}(L/K)$. We assume L/K to be unramified outside a finite set of places S of K that also contains all archimedean places. We also take an auxiliary finite set of primes T as usual when we study the Stark conjecture. Then, in §2, we shall use the leading terms at an arbitrary integer point j of the S-truncated T-modified L-series $L_{K,S,T}(\chi, s)$ of complex linear characters χ of G to unconditionally define canonical generalized Stark elements (see Definition 2.9).

We will refer to $w := -2j$ as the 'weight' of these generalized Stark elements (this corresponds to the fact that w is the weight of the associated motive $h^0(\mathrm{Spec}\, L)(j))$ and define their 'rank' in terms of the precise exterior power of the arithmetic module in which they are constructed. In particular, in weight zero, our construction will recover the classical theory of Rubin-Stark elements for L/K, and hence in weight zero and rank one it recovers the original constructions of Stark.

In §3 we shall then formulate (as Conjecture 3.6) a natural extension of the 'refined Rubin-Stark Conjecture' to encompass generalized Stark elements. Assuming that S contains all p-adic places of K for an odd prime p, this conjecture predicts that Stark elements of weight $w = -2j$ (and of appropriate rank) over L should precisely determine the initial Fitting ideal of the (T-modified) étale cohomology groups $H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$, regarded as $\mathbb{Z}_p[G]$-modules in the natural way. In this way, the approach developed here is a natural generalization of the theory obtained in [5].

In §4 we shall then obtain concrete evidence in support of Conjecture 3.6. In particular, we shall prove the conjecture for all absolutely abelian fields and, modulo Iwasawa's conjecture on the vanishing of cyclotomic p-adic μ-invariants, we shall also prove it for the minus part of CM-extensions of totally real fields (see Theorems 4.1 and 4.4 respectively).

We remark that the definition of generalized Stark elements requires us to construct a general formalism of 'period-regulator isomorphisms' in a precise and coherent fashion.

In addition, we believe that the prediction of Conjecture 3.6 has some intrinsic interest since the cohomology groups $H^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ are known to encompass many natural arithmetic modules (as w, and hence j, varies).

For these reasons, we regard the formulation of Conjecture 3.6 and the results of Theorems 4.1 and 4.4 as the main achievements of this article.

1.1.3. However, an additional motivation for developing the general formalism described here is that, in a sequel to this article, it will allow the first and third authors to formulate a precise integral family of *congruence relations* between generalized Stark elements of *different weights*.

In this way one is able to study generalized Stark elements as *p-adic families* and hence bring a new and useful perspective to the theory developed here.

In particular, one can show the congruence conjecture that is formulated in the sequel gives a simultaneous (conjectural) extension to general number fields of a wide variety of well-known conjectures and results in the literature including the classical congruences of Kummer concerning Bernoulli numbers, the theorem of Coleman and Ihara, the results of Beilinson and Huber-Wildeshaus on the cyclotomic elements of Deligne-Soulé, the p-adic Stark conjecture at $s = 1$ and the p-adic Beilinson conjecture formulated by Besser, Buckingham, de Jeu and Roblot.

These various concrete connections will then, in turn, allow one to derive strong evidence, both theoretical and numerical, in support of the general conjectural framework that is developed in the present article.

1.1.4. We make two general observations concerning the approach that is adopted here.

Firstly, we remark that a particular feature of this article is a systematic use of the notion of '*exterior power biduals*', as defined in §3.1. In fact, our approach will show that such modules constitute the natural habitat of generalized Stark elements. In addition, in the sequel to this article, the formalism of exterior power biduals plays a pivotal role in the natural formulation of a family of precise *congruence relations* between Stark elements of differing weights.

Secondly, to help set the context, we explain the difference between the generalized Stark elements that are defined here and the very general notion of arithmetic 'zeta element' that originates with Kato.

We recall that zeta elements of the latter sort are, by definition, elements in the determinant modules of perfect complexes and, as such, can be seen to belong to a very abstract and *'ideal' world* (see, for example, Definition 4.3).

On the other hand, the Stark elements that we define here are concrete generalizations of (exterior products of) algebraic units. In particular, they belong to the *'usual' arithmetic world* and so it is much easier both to extract concrete information from them and to use them to directly study problems in algebraic number theory.

Thus, whilst both Stark elements and zeta elements are related to the leading terms of L-series, there are very significant differences and it seems quite remarkable that there should be concrete links between them. (For example, in the present article, the reader will find that such links play a key role in our arguments of §4.)

Nevertheless, it should be stressed that the development, and interest, of the theory of generalized Stark elements is, in most respects, completely independent of any possible connections to zeta elements.

Moreover, the strong advantage of our theory is that it provides a very concrete, and explicit, approach that will be seen to unify, illuminate and extend a wide range of classical conjectures and problems that have been studied in the literature.

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1.2. Notation

For the reader's convenience we end the introduction by collecting together details concerning notation and conventions that are used in the sequel.

1.2.1. *Algebra* Let E be a field of characteristic 0. For any abelian group A, we denote $E \otimes_{\mathbb{Z}} A$ by EA. If A is a $\mathbb{Q}$-vector space, we sometimes denote $E \otimes_{\mathbb{Q}} A$ also by EA. Similarly, if E is an extension of $\mathbb{Q}_p$ (p is a prime number) and A is a $\mathbb{Z}_p$-module, we denote $E \otimes_{\mathbb{Z}_p} A$ and $E \otimes_{\mathbb{Q}_p} A$ also by EA. For any integer m, we denote A/mA simply by A/m.

For a commutative ring R and an R-module M we set

$$M^* := \mathrm{Hom}_R(M, R).$$

If M is a free R-module with basis $\{b_1, \ldots, b_r\}$, then for each i with $1 \leq i \leq r$ we write b_i^* for the homomorphism $M \to R$ that sends b_j to 1 if $i = j$ and to 0 if $i \neq j$.

For any field E, the absolute Galois group of E is denoted by G_E. Let $c \in G_{\mathbb{R}}$ denote the complex conjugation. For a $\mathbb{Z}[G_{\mathbb{R}}]$-module M, let $M^{\pm}$ be the submodule $\{a \in M \mid c \cdot a = \pm a\}$ of M. We also use the idempotent

$$e^{\pm} := \frac{1 \pm c}{2}$$

of $\mathbb{Z}[\frac{1}{2}][G_{\mathbb{R}}]$ and the decomposition $M = M^+ \oplus M^-$ with $M^{\pm} = e^{\pm}M$ for any $\mathbb{Z}[\frac{1}{2}][G_{\mathbb{R}}]$-module M.

1.2.2. *Arithmetic* Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For any non-negative integer m, we denote by μ_m the subgroup of all m-th roots of unity in $\overline{\mathbb{Q}}^{\times}$. As usual, we denote μ_{p^n} (p is a prime number) by $\mathbb{Z}/p^n(1)$, and $\varprojlim_n \mu_{p^n}$ by $\mathbb{Z}_p(1)$. For any integer j, $\mathbb{Z}_p(j)$ and $\mathbb{Q}_p(j)$ are defined in the usual way.

For a number field K, i.e. a finite extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$, we write $S_\infty(K)$, $S_{\mathbb{C}}(K)$ and $S_p(K)$ for the set of archimedean, complex and p-adic places of K respectively. We write S_∞ for $S_\infty(K)$ if there is no danger of confusion. The ring of integers of K is denoted by $\mathcal{O}_K$. For a finite set S of places of K, the ring of S-integers of K is denoted by $\mathcal{O}_{K,S}$. If L is a finite extension of K, then the set of places of K which ramify in L is denoted by $S_{\mathrm{ram}}(L/K)$ and the set of places of L lying above any given set of places S of K is denoted by S_L. The ring of S_L-integers of L is denoted by $\mathcal{O}_{L,S}$ instead of $\mathcal{O}_{L,S_L}$.

Let L/K be a finite abelian extension with Galois group G. Let S and T be finite disjoint sets of places of K such that

$$S_\infty(K) \cup S_{\rm ram}(L/K) \subset S.$$

Then, for a character $\chi \in \widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$, the S-truncated T-modified L-function is defined by

$$L_{K,S,T}(\chi, s) := \prod_{v \in T}(1-\chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s}) \prod_{v \notin S}(1-\chi(\mathrm{Fr}_v)\mathrm{N}v^{-s})^{-1} \;(\mathrm{Re}(s) > 1)$$

where $\mathrm{Fr}_v \in G$ is the Frobenius automorphism at a place of L above v, and $\mathrm{N}v$ is the cardinality of the residue field $\kappa(v)$ of v. The function $L_{K,S,T}(\chi, s)$ continues meromorphically to the whole complex plane and its leading term at an integer j is denoted by $L^*_{K,S,T}(\chi, j)$. The S-truncated T-modified L-function for L/K is defined by setting

$$\theta_{L/K,S,T}(s) := \sum_{\chi \in \widehat{G}} L_{K,S,T}(\chi^{-1}, s)e_\chi \quad \text{with} \quad e_\chi := \frac{1}{\#G}\sum_{\sigma \in G}\chi(\sigma)\sigma^{-1}$$

and has leading term at $s = j$ equal to

$$\theta^*_{L/K,S,T}(j) := \sum_{\chi \in \widehat{G}} L^*_{K,S,T}(\chi^{-1}, j)e_\chi \in \mathbb{C}[G]^\times.$$

When $T = \emptyset$, we omit it from notations (so we denote $L_{K,S,\emptyset}(\chi, s)$ by $L_{K,S}(\chi, s)$, for example). Note that

$$\theta_{L/K,S,T}(s) = \delta_{L/K,T}(s) \cdot \theta_{L/K,S}(s)$$

with $\delta_{L/K,T}(s) := \prod_{v \in T}(1 - \mathrm{N}v^{1-s}\mathrm{Fr}_v^{-1})$.

In each degree i the étale cohomology group $H^i_{et}(\mathrm{Spec}\,\mathcal{O}_{L,S}, \cdot)$ will be denoted by $H^i(\mathcal{O}_{L,S}, \cdot)$.

§2. Generalized Stark elements

2.1. The general set up

We consider a finite abelian extension L/K of number fields. Set $G := \mathrm{Gal}(L/K)$, and fix an *odd* prime number p. In the study of Stark elements with arbitrary weight, infinite places play an important role, so we first introduce some notations of modules and elements which are related to infinite places.

For each (finite or infinite) place w of L, we fix an algebraic closure $\overline{L}_w$ of L_w and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{L}_w$. From this, we regard G_{L_w}

as a subgroup of G_L, and the localization map of Galois cohomology $H^i(L,\cdot) \to H^i(L_w,\cdot)$ is defined by the restriction map. Also, for each place w in $S_\infty(L)$, we identify $\overline{L}_w$ with $\mathbb{C}$. For each integer j we set

$$S_\infty^j(L) := \begin{cases} S_\infty(L) & \text{if } j \text{ is even,} \\ S_{\mathbb{C}}(L) & \text{if } j \text{ is odd,} \end{cases}$$

and define $Y_L(j)$ by

$$Y_L(j) := \bigoplus_{w \in S_\infty(L)} H^0(L_w, \mathbb{Z}_p(j)).$$

Therefore, we have $Y_L(j) = \bigoplus_{w \in S_\infty^j(L)} \mathbb{Z}_p(j)$.

In particular, setting $\xi := (e^{2\pi\sqrt{-1}/p^n})_n \in H^0(\mathbb{C}, \mathbb{Z}_p(1))$ one obtains a $\mathbb{Z}_p$-basis $\{w(j)\}_{w \in S_\infty^j(L)}$ of $Y_L(j)$, which is defined by $w(j) = (w(j)_{w'})_{w'}$ where

$$w(j)_{w'} := \begin{cases} \xi^{\otimes j} & \text{if } w' = w, \\ 0 & \text{if } w' \neq w. \end{cases}$$

Next we note that the complex conjugation c in $G_{\mathbb{R}}$ acts on the Betti cohomology

$$H_L(j) := H_B^0(\operatorname{Spec} L(\mathbb{C}), \mathbb{Q}(j)) = \bigoplus_{\iota : L \hookrightarrow \mathbb{C}} (2\pi\sqrt{-1})^j \mathbb{Q}$$

by $c \cdot (a_\iota)_\iota := (c \cdot a_\iota)_{c \circ \iota}$ for each a_ι in $(2\pi\sqrt{-1})^j\mathbb{Q}$ and we set

$$H_L(j)^+ := e^+ H_L(j).$$

Then the natural decomposition $\mathbb{C} = \mathbb{R}(j) \oplus \mathbb{R}(j-1)$ induces an isomorphism of $\mathbb{R}[G \times G_{\mathbb{R}}]$-modules

$$\mathbb{C} \otimes_{\mathbb{Q}} L \simeq \bigoplus_{\iota : L \hookrightarrow \mathbb{C}} (\mathbb{R}(j) \oplus \mathbb{R}(j-1)) = \mathbb{R}H_L(j) \oplus \mathbb{R}H_L(j-1).$$

By taking the $G_{\mathbb{R}}$-invariant part of this isomorphism, we therefore obtain a canonical isomorphism of $\mathbb{R}[G]$-modules

$$\mathbb{R} \otimes_{\mathbb{Q}} L \simeq \mathbb{R}H_L(j)^+ \oplus \mathbb{R}H_L(j-1)^+. \tag{1}$$

For each embedding $\iota' : L \hookrightarrow \mathbb{C}$ we define $\iota'_j = (\iota'_{j,\iota})_\iota$ in $H_L(j)$ by setting

$$\iota'_{j,\iota} := \begin{cases} (2\pi\sqrt{-1})^j & \text{if } \iota = \iota', \\ 0 & \text{if } \iota \neq \iota'. \end{cases}$$

Then, if for each place w in $S_\infty(L)$ we write $\iota_w : L \hookrightarrow \mathbb{C}$ for the embedding induced by the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{L}_w = \mathbb{C}$, we obtain an isomorphism of $\mathbb{Q}_p[G]$-modules

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} Y_L(j) \xrightarrow{\sim} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H_L(j)^+ \tag{2}$$

that sends each element $w(j)$ to $(1+c)\iota_{w,j}$.

For an idempotent ε of $\mathbb{Z}_p[G]$, we will first define a non-negative integer r_j^ε.

For each place v in $S_\infty(K)$ we fix a place w_v in $S_\infty(L)$ that lies above v, and write $S_\infty(L)/G$ for the set $\{w_v \mid v \in S_\infty(K)\}$.

For any idempotent ε in $\mathbb{Z}_p[G]$ we then set

$$W_j^\varepsilon := \left\{ w \in S_\infty^j(L) \cap (S_\infty(L)/G) \mid \varepsilon \cdot w(-j) \neq 0 \right\}$$

and finally define

$$r_j^\varepsilon := \# W_j^\varepsilon.$$

The following observation will be useful in the sequel.

Lemma 2.1. *If ε is a primitive idempotent of $\mathbb{Z}_p[G]$ (so that the ring $\mathbb{Z}_p[G]\varepsilon$ is local), then $\varepsilon Y_L(-j)$ is a free $\mathbb{Z}_p[G]\varepsilon$-module of rank r_j^ε with basis $\{\varepsilon \cdot w(-j) \mid w \in W_j^\varepsilon\}$.*

Proof. For each place w in $S_\infty^j(L)$ the $\mathbb{Z}_p[G]$-submodule of $Y_L(-j)$ that is generated by $w(-j)$ is isomorphic to $\mathbb{Z}_p[G/G_w]$, where G_w denotes the decomposition subgroup of w. Thus, since p is odd and $\# G_w$ divides 2, it follows that $Y_L(-j)$ is a projective $\mathbb{Z}_p[G]$-module.

Hence, if ε is a primitive idempotent, then $\mathbb{Z}_p[G]\varepsilon \cdot w(-j)$ is either zero or a free $\mathbb{Z}_p[G]\varepsilon$-module of rank one and so the decomposition

$$\varepsilon Y_L(-j) = \bigoplus_{w \in S_\infty^j(L) \cap S_\infty(L)/G} \mathbb{Z}_p[G]\varepsilon \cdot w(-j)$$

implies that $\varepsilon Y_L(-j)$ is free with basis $\{\varepsilon \cdot w(-j) \mid w \in W_j^\varepsilon\}$. Q.E.D.

Before making the next definition, we note that the algebra $\mathbb{Z}_p[G]$ is semilocal and hence that each of its idempotents can be written as a sum of primitive idempotents.

Definition 2.2. Let ε be an idempotent of $\mathbb{Z}_p[G]$ and $\varepsilon = \sum_a \varepsilon_a$ its decomposition as a sum of primitive idempotents. Then we shall say that ε has *uniform rank* with respect to the integer j if the value of $r_j^{\varepsilon_a}$ is independent of the idempotent ε_a.

Remark 2.3. If the idempotent ε has uniform rank, then Lemma 2.1 implies that $\varepsilon Y_L(-j)$ is a free $\mathbb{Z}_p[G]\varepsilon$-module of rank r_j^ε and has as basis the set $\{\varepsilon \cdot w(-j) \mid w \in W_j^\varepsilon\}$.

Remark 2.4. One knows that any idempotent of $\mathbb{Z}_p[G]$ must belong to the subring $\mathbb{Z}_p[H]$, with H the maximal subgroup of G of order prime to p. This fact is not difficult to prove, and follows, for example, from the result of D. B. Coleman in [7].

Example 2.5. Suppose K is totally real and L is CM and write c for the complex conjugation in G. For each integer j we obtain idempotents of $\mathbb{Z}_p[G]$ by setting

$$e_j^{\pm} := (1 \pm (-1)^j c)/2$$

and we abbreviate $W_j^{e_j^{\pm}}$ to $W_j^{\pm}$. Then each idempotent $e_j^{\pm}$ has uniform rank and $r_j^{e_j^{\pm}}$ can be computed explicitly in the following way.

(i) If $\varepsilon = e_j^+$, then $\varepsilon \cdot w(-j) = w(-j)$ for each w in $S_\infty^j(L) \cap S_\infty(L)/G = S_\infty(L)/G$ and so we have $W_j^+ = S_\infty(L)/G$ and $r_j^\varepsilon = \#S_\infty(L)/G = \#S_\infty(K) = [K : \mathbb{Q}]$.

(ii) If $\varepsilon = e_j^-$, then $\varepsilon \cdot w(-j) = 0$ for each w in $S_\infty^j(L)$ so W_j^- is empty and $r_j^\varepsilon = 0$.

2.2. The period-regulator isomorphisms

In this section we assume that the idempotent ε has uniform rank with respect to an integer j (see Definition 2.2 and Remark 2.3), and will define an associated idempotent ε_j.

In the sequel we fix a finite set S of places of K and assume that

$$S_\infty(K) \cup S_p(K) \cup S_{\mathrm{ram}}(L/K) \subseteq S.$$

We also fix (and do not explicitly mention) an isomorphism of fields $\mathbb{C} \simeq \mathbb{C}_p$.

We write $\widehat{G}^\varepsilon$ for the subset of $\widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$ comprising characters χ for which $\varepsilon \cdot e_\chi \neq 0$. We define a subset of $\widehat{G}^\varepsilon$ by setting

$$\widehat{G}_j^\varepsilon := \left\{\chi \in \widehat{G}^\varepsilon \mid \dim_{\mathbb{C}_p}\left(e_\chi \mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))\right) = r_j^\varepsilon\right\}$$

where $H^i(\mathcal{O}_{L,S}, \cdot) := H^i_{et}(\mathrm{Spec}\,\mathcal{O}_{L,S}, \cdot)$ is the étale cohomology group. We define an idempotent ε_j of $\mathbb{Q}_p[G]\varepsilon$ by setting

$$\varepsilon_j := \sum_{\chi \in \widehat{G}_j^\varepsilon} e_\chi.$$

Remark 2.6. Lemma 4.2(ii) below implies that for each $\chi \in \widehat{G}^{\varepsilon}$ one has

$$\chi \in \widehat{G}^{\varepsilon}_j \Longleftrightarrow \begin{cases} e_\chi \mathbb{C}_p H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j)) \text{ vanishes,} & \text{if } j \neq 1, \\ e_\chi (\mathbb{C}_p \oplus \mathbb{C}_p H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p)) \text{ vanishes,} & \text{if } j = 1. \end{cases}$$

By using this description one can deduce the following facts.

(i) If $j < 0$, then $\widehat{G}^{\varepsilon}_j = \widehat{G}^{\varepsilon}$ (by Soulé [13, Th. 10.3.27]) and so $\varepsilon_j = \varepsilon$.

(ii) If $j = 0$, then by class field theory one knows that there is a canonical isomorphism $H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p X_{L,S \setminus S_\infty}$ where for any finite set Σ of places of K we write $X_{L,\Sigma}$ for the kernel of the homomorphism $\bigoplus_{w \in \Sigma_L} \mathbb{Z}_p w \to \mathbb{Z}_p$ sending each w to 1. The set $\widehat{G}^{\varepsilon}_0$ is therefore equal to $\{\chi \in \widehat{G}^{\varepsilon} \mid e_\chi \mathbb{C}_p X_{L,S\setminus S_\infty} = 0\}$.

Upon combining this description with the natural exact sequence

$$0 \to X_{L,S\setminus S_\infty} \to X_{L,S} \to Y_L(0) \to 0,$$

and the fact that for any character $\chi \in \widehat{G}^{\varepsilon}$ one has both

$$r^{\varepsilon}_0 = \dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p Y_L(0))$$

(as follows directly from the definition of r^{ε}_0) and

$$\mathrm{ord}_{s=0} L_{K,S}(\chi, s) = \dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p X_{L,S})$$

(by [19, Chap. I, Prop. 3.4]), one finds that

$$\widehat{G}^{\varepsilon}_0 = \big\{\chi \in \widehat{G}^{\varepsilon} \mid \mathrm{ord}_{s=0} L_{K,S}(\chi, s) = r^{\varepsilon}_0\big\}.$$

(iii) If $j = 1$, then Leopoldt's Conjecture for L is equivalent to the vanishing of $H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p)$ and hence implies that

$$\widehat{G}^{\varepsilon}_1 = \{\chi \in \widehat{G}^{\varepsilon} \mid \chi \neq \mathbf{1}\},$$

where we write $\mathbf{1}$ for the trivial character of G. So $\varepsilon_1 = \varepsilon(1 - e_{\mathbf{1}})$ if we assume Leopoldt's Conjecture.

(iv) If $j > 1$, then Schneider's Conjecture [17] for L is equivalent to the vanishing of $H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$ and hence it implies $\widehat{G}^{\varepsilon}_j = \widehat{G}^{\varepsilon}$ and $\varepsilon_j = \varepsilon$.

In the remainder of this section we define for each integer j a canonical isomorphism of $\mathbb{C}_p[G]$-modules

$$\lambda_j : \varepsilon_j \mathbb{C}_p {\bigwedge}^{r_j^\varepsilon}_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \xrightarrow{\sim} \varepsilon_j \mathbb{C}_p {\bigwedge}^{r_j^\varepsilon}_{\mathbb{Z}_p[G]} Y_L(-j).$$

2.2.1. *The case $j < 0$* In this case we rely on the existence of a canonical Chern character isomorphism

$$\mathrm{ch}_j : K_{1-2j}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)),$$

where we write $K_*(-)$ for Quillen's higher algebraic K-theory functor.

In fact, the existence of such an isomorphism was initially conjectured by Quillen and Lichtenbaum and subsequently shown by Suslin to follow as a consequence of the conjecture of Bloch and Kato relating Milnor K-theory to étale cohomology and then, following fundamental work of Voevodsky and Rost, Weibel completed the proof of the Bloch-Kato Conjecture in [20].

In this case one also has $\varepsilon_j = \varepsilon$ (by Remark 2.6(i)) and we define λ_j to be the r_j^ε-th exterior power of the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\begin{aligned} \varepsilon \mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) &\xrightarrow{\sim} \varepsilon \mathbb{C}_p K_{1-2j}(\mathcal{O}_L) \\ &\xrightarrow{\sim} \varepsilon \mathbb{C}_p H_L(-j)^+ \\ &\xrightarrow{\sim} \varepsilon \mathbb{C}_p Y_L(-j), \end{aligned}$$

where the first map is induced by the inverse of the isomorphism ch_j^{-1}, the second by (-1)-times the Borel regulator map

$$b_j : \mathbb{R}K_{1-2j}(\mathcal{O}_L) \xrightarrow{\sim} \mathbb{R}H_L(-j)^+$$

and the third by the isomorphism in (2).

2.2.2. *The case $j = 0$* We note that $H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ is identified with $\mathbb{Z}_p \mathcal{O}_{L,S}^\times$ via Kummer theory and we define λ_0 to be the r_0^ε-th exterior power of the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\varepsilon_0 \mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) = \varepsilon_0 \mathbb{C}_p \mathcal{O}_{L,S}^\times \xrightarrow{\sim} \varepsilon_0 \mathbb{C}_p X_{L,S} \xrightarrow{\sim} \varepsilon_0 \mathbb{C}_p Y_L(0) \tag{3}$$

where the first map is the restriction of the Dirichlet regulator (sending each a in $\mathcal{O}_{L,S}^\times$ to $-\sum_{w \in S_L} \log |a|_w w$) and the second isomorphism is a natural isomorphism obtained by $\varepsilon_0 \mathbb{C}_p X_{L,S \setminus S_\infty} = 0$, which follows from the definition of ε_0 (see Remark 2.6(ii)).

2.2.3. *The case $j = 1$* We write $\Gamma_{L,S}$ for the Galois group of the maximal abelian pro-p extension of L unramified outside S. For a p-adic place w of L, we write $U^1_{L_w}$ for the pro-p completion of $\mathcal{O}^\times_{L_w}$.

Then the module $H^1(\mathcal{O}_{L,S},\mathbb{Q}_p)$ identifies with $\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{L,S},\mathbb{Q}_p)$ and so, by combining the global class field theory with Remark 2.6(iii), one obtains a canonical short exact sequence of $\mathbb{C}_p[G]$-modules

(4)

$$\begin{array}{ccccc} \varepsilon_1\mathbb{C}_pH^1(\mathcal{O}_{L,S},\mathbb{Q}_p) & \hookrightarrow & \varepsilon_1\left(\bigoplus_{w\in S_p(L)}\mathbb{C}_pU^1_{L_w}\right)^* & \twoheadrightarrow & \varepsilon_1(\mathbb{C}_p\mathcal{O}^\times_L)^* \\ & & {\scriptstyle -\exp_p^*}\downarrow\simeq & & \uparrow\simeq \\ & & \varepsilon_1(\mathbb{C}_p\otimes_{\mathbb{Q}}L)^* & & \varepsilon_1(\mathbb{C}_pH_L(0)^+)^*. \end{array}$$

Here we write $(-)^*$ for $\mathbb{C}_p$-linear dual and identify $\bigoplus_{w\in S_p(L)}U^1_{L_w}$ with the direct sum $\bigoplus_{v\in S_p(K)}\mathbb{Z}_p[G]\otimes_{\mathbb{Z}_p[G_v]}U^1_{L_w}$, where G_v denotes the decomposition subgroup in G of any place of L above v, in order to regard it as a $\mathbb{Z}_p[G]$-module. In addition, the first vertical isomorphism is induced by the linear dual of (-1)-times the p-adic exponential map homomorphisms $L_w\to\mathbb{Q}_pU^1_{L_w}$ for w in $S_p(L)$ and the second by the linear dual of the isomorphism $\mathbb{C}_p\mathcal{O}^\times_L\simeq\mathbb{C}_pX_{L,S_\infty}$ induced by the Dirichlet regulator map and the fact that $\varepsilon_1\mathbb{Q}_pX_{L,S_\infty}$ is equal to $\varepsilon_1\mathbb{Q}_pY_L(0)$ and hence isomorphic to $\varepsilon_1\mathbb{Q}_pH_L(0)^+$ by (2).

Abbreviating $\det_{\mathbb{C}_p[G]}(-)$ to $\mathrm{D}(-)$, we then define λ_1 to be the composite isomorphism of $\mathbb{C}_p[G]$-modules

$$\begin{aligned} &\varepsilon_1\mathbb{C}_p{\bigwedge}^{r_1^\varepsilon}_{\mathbb{Z}_p[G]}H^1(\mathcal{O}_{L,S},\mathbb{Z}_p) \\ &=\varepsilon_1\mathrm{D}(\mathbb{C}_pH^1(\mathcal{O}_{L,S},\mathbb{Z}_p)) \\ &\simeq\varepsilon_1\left(\mathrm{D}((\mathbb{C}_p\otimes_{\mathbb{Q}}L)^*)\otimes_{\mathbb{C}_p[G]}\mathrm{D}^{-1}((\mathbb{C}_pH_L(0)^+)^*)\right) \\ &\simeq\varepsilon_1\mathrm{D}((\mathbb{C}_pH_L(1)^+)^*) \\ &\simeq\varepsilon_1\mathrm{D}((\mathbb{C}_pY_L(1))^*) \\ &\simeq\varepsilon_1\mathbb{C}_p{\bigwedge}^{r_1^\varepsilon}_{\mathbb{Z}_p[G]}Y_L(-1). \end{aligned}$$

Here the first isomorphism is the canonical isomorphism induced by (4), the second is induced by the linear dual of (1) (with $j=1$), the third by (2) and the last by the canonical identification $\mathrm{Hom}_{\mathbb{Z}_p}(Y_L(1),\mathbb{Z}_p)\simeq Y_L(-1)$.

2.2.4. *The case $j > 1$* In this case the space $\varepsilon_jH^2(\mathcal{O}_{L,S},\mathbb{Q}_p(1-j))$ vanishes (see Remark 2.6) and so the local and global duality theorems combine to give a canonical short exact sequence of $\mathbb{C}_p[G]$-modules

$$\begin{array}{ccccc} \Omega(1-j) & \hookrightarrow & \varepsilon_j \left(\bigoplus_{w\in S_p(L)} \mathbb{C}_p H^1(L_w, \mathbb{Z}_p(j))\right)^* & \twoheadrightarrow & \Omega(j)^* \\ & & {\scriptstyle -\mathrm{syn}_p}\downarrow \simeq & & \uparrow \simeq \\ & & \varepsilon_j(\mathbb{C}_p \otimes_{\mathbb{Q}} L)^* & & \varepsilon_j \mathbb{C}_p H_L(j-1)^{+,*} \end{array}$$

where we set $\Omega(a) := \varepsilon_j \mathbb{C}_p H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(a))$ for each integer a. In this diagram we also again write $(-)^*$ for $\mathbb{C}_p$-linear dual and identify

$$\bigoplus_{w\in S_p(L)} H^1(L_w, \mathbb{Z}_p(j))$$

with the direct sum

$$\bigoplus_{v\in S_p(K)} \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G_v]} H^1(L_w, \mathbb{Z}_p(j))$$

in order to regard it as a $\mathbb{Z}_p[G]$-module. In addition, the second vertical homomorphism is induced by the dual of $-b_{1-j} \circ \mathrm{ch}_{1-j}^{-1}$ and the first isomorphism is induced by the linear duals for each w in $S_p(L)$ of (-1)-times the canonical composite homomorphisms

$$L_w \to H^1_{\mathrm{syn}}(\mathcal{O}_{L_w}, j) \to H^1(L_w, \mathbb{Q}_p(j))$$

involving syntomic cohomology that are discussed by Besser in [1, (5.3) and Cor. 9.10].

We then define λ_j to be the isomorphism of $\mathbb{C}_p[G]$-modules obtained from the above diagram in just the same way that λ_1 is obtained from (4).

Remark 2.7. In [1, Prop. 9.11] Besser proves that for w in $S_p(L)$ the composite homomorphism $L_w \to H^1_{\mathrm{syn}}(\mathcal{O}_{L_w}, j) \to H^1(L_w, \mathbb{Q}_p(j))$ used above coincides with the exponential map of Bloch and Kato for $\mathbb{Q}_p(j)$ over L_w. In this way the definition of λ_j for $j > 1$ is naturally analogous to the definition of λ_1.

Remark 2.8. A closer analysis of the discussions used to define λ_j for $j > 0$ shows that, in this case, if $\varepsilon Y_L(1-j)$ vanishes, then $\varepsilon_j = \varepsilon$.

2.3. The definition of generalized Stark elements

Definition 2.9. Fix an integer j and an idempotent ε that has uniform rank with respect to j (see Definition 2.2 and Remark 2.3). Also fix finite sets S and T of places of K satisfying the following hypotheses:

- $S_\infty(K) \cup S_p(K) \cup S_{\rm ram}(L/K) \subset S$;
- $S \cap T = \emptyset$;
- $T = \emptyset$ if $j = 1$.

Then the 'Stark element of rank r_j^ε and weight $-2j$' for $(L/K, S, T, \varepsilon)$ is the unique element $\eta^\varepsilon_{L/K,S,T}(j)$ of $\varepsilon_j \mathbb{C}_p \bigwedge^{r_j^\varepsilon}_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ that satisfies

$$\lambda_j(\eta^\varepsilon_{L/K,S,T}(j)) = \varepsilon_j \theta^*_{L/K,S,T}(j) \cdot \bigwedge\nolimits_{w \in W_j^\varepsilon} w(-j)$$

in $\varepsilon_j \mathbb{C}_p \bigwedge^{r_j^\varepsilon}_{\mathbb{Z}_p[G]} Y_L(-j)$.

Remark 2.10. It is natural to regard $\eta^\varepsilon_{L/K,S,T}(j)$ to be of weight $-2j$ since it is associated to the motive $h^0(\mathrm{Spec}\, L)(j)$.

Example 2.11. Definition 2.9 generalizes the classical notion of Rubin-Stark element introduced by Rubin in [15]. In fact, we have

$$\varepsilon_0 \theta^*_{L/K,S,T}(0) = \varepsilon \cdot \lim_{s \to 0} s^{-r_0^\varepsilon} \theta_{L/K,S,T}(s)$$

by Remark 2.6(ii) and [19, Chap. I, Prop. 3.4] and so $\eta^\varepsilon_{L/K,S,T}(0)$ coincides with (the 'ε-component' of) the Rubin-Stark element for the data $(L/K, S, T, W_0^\varepsilon)$.

The following proposition is a natural analogue of [15, Prop. 6.1].

Proposition 2.12. *Suppose that* $(L'/K, S', T', \varepsilon')$ *is another collection of data as in Definition* 2.9 (*with respect to* j) *for which all of the following properties are satisfied*: $L \subset L'$, $S \subset S'$, $T \subset T'$, *with* $G' := \mathrm{Gal}(L'/K)$ *the natural surjection* $\mathbb{Z}_p[G'] \to \mathbb{Z}_p[G]$ *sends* ε' *to* ε *and* W_j^ε *is the set of places of* L *obtained by restricting places in* $W_j^{\varepsilon'}$.

Then $r_j^{\varepsilon'} = r_j^\varepsilon =: r$ *and the homomorphism*

$$\varepsilon'_j \mathbb{C}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G']} H^1(\mathcal{O}_{L',S'}, \mathbb{Z}_p(1-j)) \to \varepsilon_j \mathbb{C}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j))$$

that is induced by the corestriction map

$$\mathrm{Cor}_{L'/L} : H^1(\mathcal{O}_{L',S'}, \mathbb{Z}_p(1-j)) \to H^1(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j))$$

sends $\eta^{\varepsilon'}_{L'/K,S',T'}(j)$ *to*

$$\delta_{L/K,T' \setminus T}(j) \cdot \left(\prod_{v \in S' \setminus S} (1 - \mathrm{N}v^{-j} \mathrm{Fr}_v^{-1}) \right) \cdot \eta^\varepsilon_{L/K,S,T}(j)$$

where $\delta_{L/K,T}(s) = \prod_{v \in T} (1 - \mathrm{N}v^{1-s} \mathrm{Fr}_v^{-1})$.

Proof. This follows easily from the fact that the natural surjection $\mathbb{C}[G'] \to \mathbb{C}[G]$ sends $\varepsilon'_j \theta^*_{L'/K,S',T'}(j)$ to

$$\varepsilon_j \theta^*_{L/K,S',T'}(j) = \varepsilon_j \delta_{L/K,T'\setminus T}(j) \cdot \left(\prod_{v \in S' \setminus S} (1 - \mathrm{N}v^{-j} \mathrm{Fr}_v^{-1}) \right) \cdot \theta^*_{L/K,S,T}(j).$$

Q.E.D.

§3. A Rubin-Stark Conjecture in arbitrary weight

3.1. Exterior power biduals and pairings

3.1.1. *The general case* In this section we fix a commutative ring R and a finitely generated R-module M.

In the following, the expression '$\bigwedge_R^r M^*$' will always be understood to mean the r-th exterior power of the R-module given by the linear dual

$$M^* := \mathrm{Hom}_R(M, R)$$

(which we note is in general different from $(\bigwedge_R^r M)^* = \mathrm{Hom}_R(\bigwedge_R^r M, R)$).

In particular, for non-negative integers r and s with $r \leq s$ there is a canonical pairing

$$\bigwedge\nolimits_R^s M \times \bigwedge\nolimits_R^r M^* \to \bigwedge\nolimits_R^{s-r} M$$

defined by

$$(a_1 \wedge \ldots \wedge a_s, \varphi_1 \wedge \ldots \wedge \varphi_r) \mapsto \sum_{\sigma \in \mathfrak{S}_{s,r}} \mathrm{sgn}(\sigma) \det(\varphi_i(a_{\sigma(j)}))_{1 \leq i,j \leq r} a_{\sigma(r+1)} \wedge \ldots \wedge a_{\sigma(s)},$$

where we write $\mathfrak{S}_{s,r}$ for the subset of the group $\mathfrak{S}_s$ of permutations of the set $\{1, 2, \ldots, s\}$ given by

$$\big\{\sigma \in \mathfrak{S}_s \mid \sigma(1) < \ldots < \sigma(r) \text{ and } \sigma(r+1) < \ldots < \sigma(s)\big\}.$$

We denote the image of (a, Φ) under the above pairing by $\Phi(a)$.

We also use the following construction (compare [15, §1.2]).

Definition 3.1. For each non-negative integer r the r-th exterior power bidual of M is the R-module

$$\bigcap\nolimits_R^r M := \left(\bigwedge\nolimits_R^r M^*\right)^* = \mathrm{Hom}\left(\bigwedge\nolimits_R^r \mathrm{Hom}_R(M, R), R\right).$$

Remark 3.2. The assignment $a \mapsto (\Phi \mapsto \Phi(a))$ induces a homomorphism of R-modules

$$\bigwedge\nolimits_R^r M \to \bigcap\nolimits_R^r M$$

that is, in general, neither injective nor surjective. However, in the special case that R is a field (or, more generally, a finite product of fields), this map is easily seen to be bijective for every value of r.

3.1.2. *The case of group rings* In this section we consider the above constructions in the special case that R is equal to $\mathbb{Z}_p[G]$ for a finite abelian group G.

In this setting, the following result is straightforward, but nevertheless important since it allows us to regard modules of the form $\bigcap^r_{\mathbb{Z}_p[G]} M$ as subsets of $\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G]} M$.

Proposition 3.3. *The assignment $a \mapsto (\Phi \mapsto \Phi(a))$ induces an identification*

$$\left\{ a \in \mathbb{Q}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} M \;\middle|\; \Phi(a) \in \mathbb{Z}_p[G] \text{ for every } \Phi \in \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} M^* \right\} \simeq \bigcap\nolimits^r_{\mathbb{Z}_p[G]} M.$$

Proof. There is a composite isomorphism of $\mathbb{Q}_p[G]$-modules

$$\begin{aligned} \mathbb{Q}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} M &= \bigwedge\nolimits^r_{\mathbb{Q}_p[G]} (M \otimes \mathbb{Q}_p) \\ &\simeq \bigcap\nolimits^r_{\mathbb{Q}_p[G]} (M \otimes \mathbb{Q}_p) \\ &= \mathrm{Hom}_{\mathbb{Z}_p[G]}(\bigwedge\nolimits^r_{\mathbb{Z}_p[G]} M^*, \mathbb{Q}_p[G]), \end{aligned}$$

in which the indicated isomorphism is induced by the given map (in view of the final observation in Remark 3.2) and the other two identifications are clear.

This composite isomorphism in turn restricts to give an injective homomorphism from $\bigcap^r_{\mathbb{Q}_p[G]} M$ to $\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G]} M$, whose image is characterized by the property '$\Phi(a) \in \mathbb{Z}_p[G]$ for every $\Phi \in \bigwedge^r_{\mathbb{Z}_p[G]} M^*$'. Q.E.D.

In a later argument we will also use the following observation about the functorial properties of exterior power biduals.

Lemma 3.4. *Assume that the $\mathbb{Z}_p[G]$-module M is torsion-free. Let H be a subgroup of G, denote the natural surjection $\mathbb{Q}_p[G] \to \mathbb{Q}_p[G/H]$ by π_H and write*

$$\mathrm{N}^r_H : \mathbb{Q}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} M \to \mathbb{Q}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G/H]} M^H$$

for the homomorphism that is induced by the norm map

$$M \to M^H;\ m \mapsto \sum_{\sigma\in H} \sigma\cdot m.$$

Then, for every element a *of* $\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G]} M$, *we have*

$$\pi_H\left(\left\{\Phi(a)\ \middle|\ \Phi\in {\bigwedge}^r_{\mathbb{Z}_p[G]} M^*\right\}\right) = \left\{\Psi(\mathrm{N}^r_H(a))\ \middle|\ \Psi\in {\bigwedge}^r_{\mathbb{Z}_p[G/H]} (M^H)^*\right\}.$$

Proof. This follows from [16, Rem. 2.9 and Lem. 2.10]. Q.E.D.

3.2. *T*-modified cohomology

Let j be an integer, and S and T sets of places of K as in Definition 2.9.

Let now R denote any of the rings $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Z}/p^n$ for some natural number n. Then, as T is disjoint from S, for each w in T_L there is a natural morphism of étale cohomology complexes

$$R\Gamma(\mathcal{O}_{L,S}, R(1-j)) \to R\Gamma(\kappa(w), R(1-j)).$$

We define $R\Gamma_T(\mathcal{O}_{L,S}, R(1-j))$ to be a complex that lies in an exact triangle in the derived category $D(R[G])$ of complexes of $R[G]$-modules of the form

$$(5)\quad R\Gamma_T(\mathcal{O}_{L,S}, R(1-j)) \to R\Gamma(\mathcal{O}_{L,S}, R(1-j)) \to \bigoplus_{w\in T_L} R\Gamma(\kappa(w), R(1-j)) \to,$$

where the second arrow is the diagonal map induced by the morphisms described above. In each degree i we then set

$$H^i_T(\mathcal{O}_{L,S}, R(1-j)) := H^i(R\Gamma_T(\mathcal{O}_{L,S}, R(1-j))).$$

With this definition, one has

$$H^i_T(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j)) = H^i(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$$

since $R\Gamma(\kappa(w), \mathbb{Q}_p(1-j))$ is acyclic if $j\neq 1$ and we are assuming that T is empty if $j=1$, and also

$$H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) = \ker\left(H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to \bigoplus_{w\in T_L} H^1(\kappa(w), \mathbb{Z}_p(1-j))\right)$$

where the arrow denotes the natural diagonal map.

Hence, Proposition 3.3 identifies $\bigcap^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ with a sublattice of $\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G]} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$.

Example 3.5. Kummer theory identifies $H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ with the pro-p completion of the (S,T)-unit group

$$\mathcal{O}^\times_{L,S,T} := \ker\left(\mathcal{O}^\times_{L,S} \to \bigoplus_{w \in T_L} \kappa(w)^\times\right)$$

of L that plays an important role in the classical theory of Stark units.

3.3. Statement of the conjecture

In the sequel for each non-negative integer i we write $\mathrm{Fitt}^i_{\mathbb{Z}_p[G]}(M)$ for the i-th Fitting ideal of a finitely generated $\mathbb{Z}_p[G]$-module M.

We also write I_G for the augmentation ideal of $\mathbb{Z}_p[G]$.

Conjecture 3.6. *Fix an integer j, an idempotent ε of $\mathbb{Z}_p[G]$ having uniform rank with respect to j (see Definition 2.2 and Remark 2.3), and sets of places S and T as in Definition 2.9. Assume that the module $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free and, in addition, that if $j = 1$ then ε belongs to I_G. Set $\eta = \eta^\varepsilon_{L/K,S,T}(j)$.*

(i) *One has a containment*

$$\eta \in \bigcap\nolimits^{r^\varepsilon_j}_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)).$$

(ii) *More strongly, there is an equality of ideals of $\mathbb{Z}_p[G]$*

$$\left\{\Phi(\eta) \;\middle|\; \Phi \in \bigwedge\nolimits^{r^\varepsilon_j}_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))^*\right\} = \varepsilon \cdot \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}\left(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))\right).$$

Remark 3.7. The module $H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p)$ is always $\mathbb{Z}_p$-free. If $j \neq 1$, then the module $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free if and only if the composite map

$$\varepsilon H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-j)) \to \varepsilon H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to \varepsilon \bigoplus_{w \in T_L} H^1(\kappa(w), \mathbb{Z}_p(1-j))$$

is injective, where the first map is the boundary homomorphism and the second is the natural map. If T is not empty, it contains a non p-adic place and $H^1(\kappa(w), \mathbb{Z}_p(1-j)) = H^0(\kappa(w), \mathbb{Q}_p/\mathbb{Z}_p(1-j))$, so the above map is always injective. In particular, therefore, the module $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free whenever T is non-empty.

Remark 3.8. With the assumption that $\varepsilon \in I_G$ if $j = 1$, we can use the complex $R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))[1] \oplus Y_L(-j)[-1]$ to construct an exact sequence of $\mathbb{Z}_p[G]\varepsilon$-modules of the form

$$0 \to \varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to F \to F \\ \to \varepsilon\big(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \oplus Y_L(-j)\big) \to 0$$

where F is both finitely generated and free (see §4). This sequence is a natural analogue of classical 'Tate sequences' (as discussed, for example, in [5, §2.3]) and plays a key role in our analysis.

Example 3.9. If we identify $H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ with the pro-p completion of the (S,T)-unit group $\mathcal{O}^\times_{L,S,T}$ of L (see Example 3.5) then in the case of $j = 0$ the equality (ii) in Conjecture 3.6 recovers the 'ε-component' of the pro-p completion of [5, Conj. 7.3] and thus constitutes a refinement of a range of well-known conjectures in the literature. For the same reason, in the setting of Example 2.11, the $j = 0$ case of the containment (i) in Conjecture 3.6 recovers the 'ε-component' of the pro-p completion of the Rubin-Stark Conjecture [15, Conj. B′] for the data $(L/K, S, T, W_0^\varepsilon)$.

Example 3.10. Assume that K is totally real, that L is CM and that $j \le 0$ and take ε to be the idempotent e_j^- in Example 2.5(ii).
(i) In this case the containment in Conjecture 3.6(i) is unconditionally valid. To see this, note $r_j^\varepsilon = 0$ so

$$\eta^\varepsilon_{L/K,S,T}(j) = e_j^- \theta^*_{L/K,S,T}(j) = \theta_{L/K,S,T}(j) = \delta_{L/K,T}(j)\theta_{L/K,S}(j).$$

If $T \neq \emptyset$, then $\delta_{L/K,T}(j)$ (belongs to $\mathbb{Z}_p[G]$ and) annihilates the module $H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-j))$. If T is empty, then $\varepsilon H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-j))$ vanishes by the assumed injectivity of the displayed map in Remark 3.7.

Thus the claimed containment in Conjecture 3.6(i) follows from the fact that Deligne and Ribet [9] have shown $a \cdot \theta_{L/K,S}(j)$ belongs to $\mathbb{Z}_p[G]$ for any element a of $\mathbb{Z}_p[G]$ that annihilates $H^0(L, \mathbb{Q}_p/\mathbb{Z}_p(1-j))$.
(ii) If $j < 0$, then the conjectural equality in Conjecture 3.6(ii) implies that

$$\begin{aligned} \mathbb{Z}_p[G] \cdot \theta_{L/K,S,T}(j) &= \varepsilon \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}\big(H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))\big) \\ &\subseteq \varepsilon \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}\big(H^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))\big) \\ &= \varepsilon \mathbb{Z}_p \mathrm{Fitt}^0_{\mathbb{Z}[G]}(K_{-2j}(\mathcal{O}_{L,S})). \end{aligned}$$

Here the inclusion is true since $H^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is a quotient of $H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ and the final equality because the known validity of the Quillen-Lichtenbaum Conjecture (as per the discussion at the beginning of §2.2.1) gives a canonical Chern character isomorphism $H^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \simeq \mathbb{Z}_p K_{-2j}(\mathcal{O}_{L,S})$.

Taking account of the canonical short exact sequence (due to Soulé)

$$0 \to K_{-2j}(\mathcal{O}_L) \to K_{-2j}(\mathcal{O}_{L,S}) \to \bigoplus_w K_{-2j-1}(\kappa_w) \to 0$$

where w runs over non-archimedean places of L above S and κ_w denotes the corresponding residue field, one finds that the above displayed inclusion shows that Conjecture 3.6(ii) refines the classical Coates-Sinnott Conjecture, which only predicted that $\theta_{L/K,S,T}(j)$ is contained in the annihilator in $\mathbb{Z}_p[G]$ of $\mathbb{Z}_p K_{-2j}(\mathcal{O}_L)$.

We end this section by stating some functorial properties of Conjecture 3.6.

Proposition 3.11. *Let* $(L'/K, S', T', \varepsilon')$ *be as in Proposition* 2.12.

(i) *Suppose* $S' = S$ *and* $T' = T$. *Then the validity of Conjecture* 3.6 (i), *respectively* (ii), *for* $(L'/K, S, T, \varepsilon', j)$ *implies the validity of Conjecture* 3.6 (i), *respectively* (ii), *for* $(L/K, S, T, \varepsilon, j)$.

(ii) *Suppose that* $L' = L$, $T' = T$ *and* $\varepsilon' = \varepsilon$. *Then the validity of the containment* (i) *in Conjecture* 3.6 *for* $(L/K, S, T, \varepsilon, j)$ *implies its validity for* $(L/K, S', T, \varepsilon, j)$.

(iii) *Suppose that* $L' = L$, $S' = S$ *and* $\varepsilon' = \varepsilon$. *Then the validity of the containment* (i) *in Conjecture* 3.6 *for* $(L/K, S, T, \varepsilon, j)$ *implies its validity for* $(L/K, S, T', \varepsilon, j)$.

Proof. We know that $\varepsilon' R\Gamma_T(\mathcal{O}_{L',S}, \mathbb{Z}_p(1-j))$ is a perfect complex of $\mathbb{Z}_p[G']$-modules and acyclic outside degrees one and two (see Lemma 4.2 below), and that

$$R\Gamma_T(\mathcal{O}_{L',S}, \mathbb{Z}_p(1-j)) \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G']} \mathbb{Z}_p[G] \simeq R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$$

(see [11, Prop. 1.6.5], for example). From this we see that

$$\varepsilon' H^1_T(\mathcal{O}_{L',S}, \mathbb{Z}_p(1-j))^{\mathrm{Gal}(L'/L)} \simeq \varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$$

and that

$$\varepsilon' H^2_T(\mathcal{O}_{L',S}, \mathbb{Z}_p(1-j)) \otimes_{\mathbb{Z}_p[G']} \mathbb{Z}_p[G] \simeq \varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)).$$

Noting this, claim (i) follows from Proposition 2.12 and Lemma 3.4.

Next, we show claim (ii). We have an exact sequence

$$0 \to H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to H^1_T(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j)) \to \bigoplus_{w \in (S' \setminus S)_L} \frac{H^1(L_w, \mathbb{Z}_p(1-j))}{H^1(\kappa(w), \mathbb{Z}_p(1-j))}.$$

Since the last term is $\mathbb{Z}_p$-free, we see that the restriction map

$$H^1_T(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j))^* \to H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))^*$$

is surjective. From this, we see that

$$\bigcap\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \subset \bigcap\nolimits^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S'}, \mathbb{Z}_p(1-j)).$$

Now the assertion in claim (ii) is clear by Proposition 2.12.

One can prove claim (iii) in the same way as [14, Prop. 5.3.1] by using the exact triangle (5) and

$$\bigoplus_{w|v} H^1(\kappa(w), \mathbb{Z}_p(1-j)) \simeq \mathbb{Z}_p[G]/(1 - \mathrm{N}v^{1-j}\mathrm{Fr}_v^{-1})$$

for each $v \in T$. Q.E.D.

§4. Zeta elements and evidence for Conjecture 3.6

In this section, we interpret generalized Stark elements in terms of the theory of arithmetic zeta elements and use this connection to obtain the following concrete evidence in support of Conjecture 3.6.

Theorem 4.1. *Conjecture* 3.6 *is valid in both of the following cases.*

(i) *L is an abelian extension of $\mathbb{Q}$.*

(ii) *K is totally real, L is CM, $j \le 0$, ε is the idempotent e_j^- in Example* 2.5(ii), *and the Iwasawa μ-invariant vanishes for the cyclotomic $\mathbb{Z}_p$-extension L_∞/L.*

4.1. Perfect complexes

Let $\varepsilon \in \mathbb{Z}_p[G]$ be any idempotent (we do not need to assume that it has uniform rank in this subsection). Let S and T be sets of places of K as in Definition 2.9.

With Z denoting either $\mathbb{Z}_p$ or $\mathbb{Z}/p^n$ for some natural number n we define an object of $D(Z[G]\varepsilon)$ by setting

$$C^\varepsilon_{L,S,T}(j)_Z := Z[G]\varepsilon \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} (R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))[1] \oplus Y_L(-j)[-1]).$$

The properties of these complexes that we use are recorded in the following result.

In the sequel we write $D^{\rm perf}(Z[G]\varepsilon)$ for the full triangulated subcategory of $D(Z[G]\varepsilon)$ comprising complexes that are 'perfect' (that is, isomorphic to a bounded complex of finitely generated projective $Z[G]\varepsilon$-modules).

Lemma 4.2. *The following claims are valid for all integers j.*

(i) $C^{\varepsilon}_{L,S,T}(j)_Z$ *belongs to* $D^{\rm perf}(Z[G]\varepsilon)$ *and is acyclic outside degrees* $-1, 0$ *and* 1.

(ii) *Assume that* $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ *is* $\mathbb{Z}_p$*-free if* $Z = \mathbb{Z}/p^n$ *and that* $\varepsilon \in I_G$ *if* $j = 1$. *Then we have*

$$H^i(C^{\varepsilon}_{L,S,T}(j)_Z) = \begin{cases} 0 & \text{if } i = -1, \\ \varepsilon H^1_T(\mathcal{O}_{L,S}, Z(1-j)) & \text{if } i = 0, \\ \varepsilon H^2_T(\mathcal{O}_{L,S}, Z(1-j)) \oplus \varepsilon(Y_L(-j) \otimes_{\mathbb{Z}_p} Z) & \text{if } i = 1. \end{cases}$$

Furthermore, we have a (non-canonical) isomorphism of $\mathbb{Q}_p[G]$*-modules*

$$\mathbb{Q}_p H^0(C^{\varepsilon}_{L,S,T}(j)_{\mathbb{Z}_p}) \simeq \mathbb{Q}_p H^1(C^{\varepsilon}_{L,S,T}(j)_{\mathbb{Z}_p}).$$

Proof. Since p is odd, it is well-known that $R\Gamma(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ belongs to $D^{\rm perf}(\mathbb{Z}_p[G])$ and is acyclic outside degrees zero, one and two (see, for example, [11, Prop. 1.6.5]).

Claim (i) follows from this and the fact that the complex

$$\bigoplus_{w \in T_L} R\Gamma(\kappa(w), Z(1-j))$$

that occur in the triangle (5) belongs to $D^{\rm perf}(Z[G])$ and is acyclic outside degrees zero and one.

To prove the first assertion of claim (ii) it suffices to show that the module $\varepsilon H^0_T(\mathcal{O}_{L,S}, Z(1-j))$ vanishes under the stated assumptions.

If $j \neq 1$, then the module $H^0(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ vanishes and hence also $H^0_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ vanishes.

Thus, if $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free, then the exact triangle

$$R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \stackrel{p^n}{\to} R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \to R\Gamma_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j)) \to$$

implies $\varepsilon H^0_T(\mathcal{O}_{L,S}, \mathbb{Z}/p^n(1-j))$ vanishes.

Next, we consider the case when $j = 1$. Recall that we set $T = \emptyset$ in this case. Since $\varepsilon \in I_G$ by assumption, we have

$$\varepsilon H^0_T(\mathcal{O}_{L,S}, Z) = \varepsilon H^0(\mathcal{O}_{L,S}, Z) = \varepsilon \cdot Z = 0.$$

To prove the remaining assertion of claim (ii) we write $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(j))$ for the compactly supported cohomology complex of $\mathbb{Z}_p(j)$ and note that Artin-Verdier duality combines with the triangle (5) to give a canonical exact triangle in $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w\in T_L} R\Gamma(\kappa(w),\mathbb{Z}_p(1-j)) \to C^1_{L,S,T}(j)_{\mathbb{Z}_p} \to R\,\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(j)),\mathbb{Z}_p)[-2] \to .$$

Then, since $C^{\varepsilon}_{L,S,T}(j)_{\mathbb{Z}_p}$ is acyclic outside degrees zero and one, the final assertion of claim (ii) follows from this triangle and the fact that the $\mathbb{Q}_p[G]$-equivariant Euler characteristics of both of the complexes $\bigoplus_{w\in T_L} R\Gamma(\kappa(w),\mathbb{Q}_p(1-j))$ and $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Q}_p(j))$ vanish. Q.E.D.

4.2. Zeta elements

We quickly review the definition of zeta elements in the equivariant Tamagawa number conjecture,

To do this we fix notation $L/K, G, p, S, T, j, \varepsilon$ and ε_j as in §2. We often abbreviate $C^{\varepsilon}_{L,S,T}(j)_{\mathbb{Z}_p}$ to $C^{\varepsilon}_{L,S,T}(j)$. When $\varepsilon = 1$, we omit it from notations (so we denote $C^1_{L,S,T}(j)$ by $C_{L,S,T}(j)$, for example).

The definition of ε_j combines with Lemma 4.2(ii) to imply that the complex $\mathbb{Q}_p[G]\varepsilon_j \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C_{L,S,T}(j)$ is acyclic outside degrees zero and one and that there are canonical isomorphisms

$$\varepsilon_j\mathbb{Q}_p H^i(C_{L,S,T}(j))) \simeq \begin{cases} \varepsilon_j H^1_T(\mathcal{O}_{L,S},\mathbb{Q}_p(1-j)) & \text{if } i=0,\\ \varepsilon_j\mathbb{Q}_p Y_L(-j) & \text{if } i=1. \end{cases}$$

Since these $\mathbb{Q}_p[G]\varepsilon_j$-modules are both free of rank r^{ε}_j there is a canonical 'passage to cohomology' isomorphism of $\mathbb{Q}_p[G]\varepsilon_j$-modules

$$\text{(6)}\quad \pi_j : \varepsilon_j\mathbb{Q}_p\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}(j)) \xrightarrow{\sim} \varepsilon_j\mathbb{Q}_p\left(\bigwedge\nolimits^{r^{\varepsilon}_j}_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j)) \otimes_{\mathbb{Z}_p[G]} \bigwedge\nolimits^{r^{\varepsilon}_j}_{\mathbb{Z}_p[G]} Y_L(j)\right).$$

Here we identify $Y_L(j)$ with $Y_L(-j)^*$.

Definition 4.3. The zeta element associated to the data $(L/K,S,T,\varepsilon,j)$ is the unique element $z^{\varepsilon}_{L/K,S,T}(j)$ of $\varepsilon_j\mathbb{C}_p\mathrm{det}_{\mathbb{Z}_p[G]}(C_{L,S,T}(j))$ that satisfies

$$\pi_j(z^{\varepsilon}_{L/K,S,T}(j)) = \eta^{\varepsilon}_{L/K,S,T}(j) \otimes \bigwedge\nolimits_{w\in W^{\varepsilon}_j} w(j),$$

or equivalently,

$$(\mathrm{ev}_L \circ (\lambda_j \otimes \mathrm{id}) \circ \pi_j)(z^{\varepsilon}_{L/K,S,T}(j)) = \varepsilon_j\theta^*_{L/K,S,T}(j),$$

where ev_L denotes the standard 'evaluation' isomorphism

$$\bigwedge\nolimits_{\mathbb{C}_p[G]}^{r_j^\varepsilon} \mathbb{C}_p Y_L(-j) \otimes_{\mathbb{C}_p[G]} \bigwedge\nolimits_{\mathbb{C}_p[G]}^{r_j^\varepsilon} \mathbb{C}_p Y_L(-j)^* \simeq \mathbb{C}_p[G].$$

4.3. The proof of Theorem 4.1

The main result of this section is the following.

Theorem 4.4. *If there exists a* $\mathbb{Z}_p[G]\varepsilon$*-basis* z *of* $\varepsilon\det_{\mathbb{Z}_p[G]}(C_{L,S,T}(j))$ *with*

$$\varepsilon_j z = z^\varepsilon_{L/K,S,T}(j),$$

then Conjecture 3.6 *is valid.*

However, before proving this we first use it to deduce the following result.

Corollary 4.5. *Theorem* 4.1 *is valid.*

Proof. The first point to note is that the maps that are used in the explicit definition of the isomorphism λ_j given in §2.2 coincide with the maps that occur in the statement of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}\, L)(j), \mathbb{Z}_p[G]\varepsilon)$ (as stated in [2, Conj. 4(iv)]). This fact is clear if $j \leq 1$ and follows in the case $j > 1$ from the result of Besser recalled in Remark 2.7.

Given this, and our definition of the element $z^\varepsilon_{L/K,S,T}(j)$, the latter conjecture implies the existence of a $\mathbb{Z}_p[G]\varepsilon$-basis of $\varepsilon\cdot\det_{\mathbb{Z}_p[G]}(C_{L,S,T}(j))$ with the property that is stated in Theorem 4.4.

We note that this conjecture is usually formulated without using the set T, but as noted in [5, Prop. 3.4] one can formulate a natural T-modified version of this conjecture, whose validity is independent of the choice of T.

The result of Theorem 4.1(i) now follows directly from Theorem 4.4 and the fact that if L is abelian over $\mathbb{Q}$, then the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}\, L)(j), \mathbb{Z}_p[G])$ is known to be true (by work of the first author and Greither [4], of the first author and Flach [3], and of Flach [10]).

Theorem 4.1(ii) can be proved by the same method as in [6, Cor. 3.18] by using the Iwasawa main conjecture proved by Wiles. Q.E.D.

The proof of Theorem 4.4 will now occupy the rest of this section and is motivated by the argument used to prove [5, Th. 7.5].

We assume throughout that $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free and, in addition, that ε belongs to I_G if $j = 1$.

We set $\mathcal{A} := \mathbb{Z}_p[G]\varepsilon$, $A := \mathbb{Q}_p[G]\varepsilon$, $W := W^\varepsilon_j$ and $r := r^\varepsilon_j$. We also label (and thereby order) the elements of W as $\{w_i\}_{1\leq i\leq r}$.

Then Lemma 4.2(ii) implies that $C^{\varepsilon}_{L,S,T}(j)$ is acyclic outside degrees zero and one. We can therefore choose a representative of $C^{\varepsilon}_{L,S,T}(j)$ of the form

$$F \xrightarrow{\psi} F$$

with F a free $\mathcal{A}$-module with basis $\{b_1, \ldots, b_d\}$ for some sufficiently large integer d so that the natural surjection

$$F \to \mathrm{coker}(\psi) = H^1(C^{\varepsilon}_{L,S,T}(j)) = \varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \oplus \varepsilon Y_L(-j)$$

sends b_i with $1 \leq i \leq r$ to $\varepsilon \cdot w_i(-j)$ and $\{b_{r+1}, \ldots, b_d\}$ to a set of generators of the $\mathcal{A}$-module $\varepsilon H^2_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$. See [5, §5.4] for the detail of this construction. Note that the representative chosen in loc. cit. is of the form $P \to F$ with P projective and F free, but in the present case we can identify P with F by Swan's theorem (see [8, (32.1)]). Also, note that the assumption that $\varepsilon H^1_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j))$ is $\mathbb{Z}_p$-free is needed here.

We may therefore identify $\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j))$ with $\bigwedge^d_{\mathcal{A}} F \otimes_{\mathcal{A}} \bigwedge^d_{\mathcal{A}} F^*$. With respect to this identification, any $\mathcal{A}$-basis of $\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j))$ has the form

$$z_x := x \cdot b_1 \wedge \ldots \wedge b_d \otimes b_1^* \wedge \ldots \wedge b_d^*$$

with $x \in \mathcal{A}^\times$, where we write b_i^* for the $\mathcal{A}$-linear dual of b_i.

Next we write

$$\begin{aligned}\pi'_j : \mathbb{Q}_p\det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j)) &\to \varepsilon_j \mathbb{Q}_p \det_{\mathcal{A}}(C^{\varepsilon}_{L,S,T}(j)) \\ &\xrightarrow{\sim} \varepsilon_j \bigwedge\nolimits^r_{\mathbb{Q}_p[G]} H^1_T(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))\end{aligned}$$

for the composite homomorphism of A-modules in which the first map is 'multiplication by ε_j' and the second is the composite of the isomorphism π_j in (6) and the isomorphism of A-modules

$$\varepsilon_j \mathbb{Q}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[G]} Y_L(j) \xrightarrow{\sim} A\varepsilon_j$$

that sends the element $\varepsilon_j \cdot w_1(j) \wedge \ldots \wedge w_r(j)$ to ε_j.

Then we claim that

$$\begin{aligned}(7)\qquad &\pi'_j(z_x) \\ &= (-1)^{r(d-r)} x \left(\bigwedge\nolimits_{r<i\leq d} \psi_i\right)(b_1 \wedge \ldots \wedge b_d) \\ &= (-1)^{r(d-r)} x \sum_{\sigma \in \mathfrak{S}_{d,r}} \mathrm{sgn}(\sigma) \det(\psi_i(b_{\sigma(k)}))_{r<i,k\leq d} b_{\sigma(1)} \wedge \ldots \wedge b_{\sigma(r)}\end{aligned}$$

with $\psi_i := b_i^* \circ \psi \in \mathrm{Hom}_{\mathcal{A}}(F, \mathcal{A})$ for each index i.

To show this we first check that the element

$$\left(\bigwedge\nolimits_{r<i\le d}\psi_i\right)(b_1 \wedge \ldots \wedge b_d)$$

of $\bigwedge_{\mathcal{A}}^r F$ actually lies in $\varepsilon_j \bigwedge_{\mathbb{Q}_p[G]}^r H_T^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$, regarded as a submodule of $\mathbb{Q}_p \bigwedge_{\mathcal{A}}^r F$ via the inclusion

$$\varepsilon H_T^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) = H^0(C_{L,S,T}^{\varepsilon}(j)) = \ker\psi \hookrightarrow F. \tag{8}$$

In fact, if e is any primitive idempotent of A (so that Ae is a field), then setting

$$\Psi := \bigoplus_{r<i\le d} \psi_i : e\mathbb{Q}_p F \to (Ae)^{\oplus(d-r)},$$

we can apply [5, Lem. 4.2] to the Ae-vector space $e\mathbb{Q}_pF$ and the homomorphism Ψ to deduce that the element

$$e\left(\bigwedge\nolimits_{r<i\le d}\psi_i\right)(b_1 \wedge \ldots \wedge b_d)$$

belongs to $e\bigwedge_{\mathbb{Q}_p[G]}^r H_T^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$ if Ψ is surjective and vanishes otherwise.

By then noting the equivalences

$$\Psi \text{ is surjective} \Leftrightarrow eH^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j)) = 0 \Leftrightarrow e\varepsilon_j \neq 0,$$

one can deduce that the element $\left(\bigwedge_{r<i\le d}\psi_i\right)(b_1 \wedge \ldots \wedge b_d) \in \bigwedge_{\mathcal{A}}^r F$ lies in the submodule $\varepsilon_j \bigwedge_{\mathbb{Q}_p[G]}^r H_T^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1-j))$, as claimed.

To prove (7) we can now, for each primitive idempotent e of $A\varepsilon_j$, apply the result of [5, Lem. 4.3] to the Ae-vector space $e\mathbb{Q}_pF$ and the map $\psi : e\mathbb{Q}_pF \to e\mathbb{Q}_pF$ to deduce that

$$\pi_j(ez_x) = (-1)^{r(d-r)} ex\left(\bigwedge\nolimits_{r<i\le d}\psi_i\right)(b_1 \wedge \ldots \wedge b_d) \otimes (w_1(j) \wedge \ldots \wedge w_r(j)).$$

('F_ψ' in loc. cit. corresponds to our π_j in (6).) Given this equality, the claimed equality (7) follows directly from the definition of π_j'.

Next we note that the matrix of the endomorphism ψ with respect to the basis $\{b_1, \ldots, b_d\}$ of F is $(\psi_i(b_k))_{1\le i,k\le d}$.

In particular, since Lemma 4.2(ii) implies that there is a direct sum decomposition

$$H^1(C_{L,S,T}^{\varepsilon}(j)) \simeq \varepsilon H_T^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1-j)) \oplus \varepsilon Y_L(-j)$$

and the elements $\{\varepsilon\cdot w_i(-j)\}_{1\le i\le r}$ are an $\mathcal{A}$-basis of $\varepsilon Y_L(-j)$, we deduce that

$$\psi_i = 0 \text{ for each } i \text{ with } 1\le i\le r.$$

This observation implies that the matrix $(\psi_i(b_k))_{r<i\le d,1\le k\le d}$ is a relation matrix of the $\mathcal{A}$-module $\varepsilon H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))$ and hence that $\{\det(\psi_i(b_{\sigma(k)}))_{r<i,k\le d}\}_{\sigma\in\mathfrak{S}_{d,r}}$ is a set of generators of the ideal

$$\mathrm{Fitt}^0_{\mathcal{A}}\big(\varepsilon H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))\big) = \varepsilon \mathrm{Fitt}^0_{\mathbb{Z}_p[G]}\big(H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))\big).$$

Thus, since the restriction map

$$F^*\to \varepsilon H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))^*$$

is surjective (as the cokernel of (8) is $\mathbb{Z}_p$-free) we can deduce from the equality (7) that

$$\left\{\Phi(\pi'_j(z_x)) \;\middle|\; \Phi\in\varepsilon{\bigwedge}^r_{\mathbb{Z}_p[G]} H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))^*\right\} = \varepsilon\cdot\mathrm{Fitt}^0_{\mathbb{Z}_p[G]}\big(H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1-j))\big). \tag{9}$$

Now suppose that $\varepsilon_j\cdot z_x = z^{\varepsilon}_{L/K,S,T}(j)$. Then the definition of $z^{\varepsilon}_{L/K,S,T}(j)$ implies that

$$\pi'_j(z_x) = \eta^{\varepsilon}_{L/K,S,T}(j)$$

and, given this, the result of Theorem 4.4 follows directly from the equality (9).

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